Notes about Successful Discriminator Searches

Chad E. Brown

Draft of April 17, 2020

Contents

Chapter 1

Introduction

This is just some notes about successful discriminator searches.

1.1 Definitions and Axioms

Let us first make explicit the set of definitions and axioms. We will take primitive names for the first five ordinals: 0, 1, 2, 3 and 4. Note that 0 will also be our notation for the empty set. All the axioms below obviously hold in every (reasonable) set theoretic model. The definition of $\mathsf{atleast2}$ is nonstandard, but can be proven to mean that the set in question has at least two elements.

Definition 1.1. *We define $\subseteq$ to be a binary relation such that $X \subseteq Y$ holds iff (if and only if) $\forall x. x \in X \Rightarrow x \in Y$.*

Definition 1.2. *We define* disj *to be a binary relation such that* $\mathsf{disj}\ X\ Y$ *holds iff* $X \cap Y = 0$.

Definition 1.3. *We define* $\mathsf{atleast2}$ *to be a unary predicate such that* $\mathsf{atleast2}\ X$ *holds iff* $\exists Y \in X. X \not\subseteq \wp\, Y$.

Definition 1.4. *We define* $\mathsf{atleast3}$ *to be a unary predicate such that* $\mathsf{atleast3}\ X$ *holds iff* $\exists Y \subseteq X. X \not\subseteq Y \wedge \mathsf{atleast2}\ Y$.

Definition 1.5. *We define* $\mathsf{atleast4}$ *to be a unary predicate such that* $\mathsf{atleast4}\ X$ *holds iff* $\exists Y \subseteq X. X \not\subseteq Y \wedge \mathsf{atleast3}\ Y$.

Definition 1.6. *We define* $\mathsf{atleast5}$ *to be a unary predicate such that* $\mathsf{atleast5}\ X$ *holds iff* $\exists Y \subseteq X. X \not\subseteq Y \wedge \mathsf{atleast4}\ Y$.

Definition 1.7. *We define* $\mathsf{atleast6}$ *to be a unary predicate such that* $\mathsf{atleast6}\ X$ *holds iff* $\exists Y \subseteq X. X \not\subseteq Y \wedge \mathsf{atleast5}\ Y$.

Definition 1.8. *We define* $\mathsf{atleast7}$ *to be a unary predicate such that* $\mathsf{atleast7}\ X$ *holds iff* $\exists Y \subseteq X. X \not\subseteq Y \wedge \mathsf{atleast6}\ Y$.

Definition 1.9. *We define* $\mathsf{exactly2}$ *to be a unary predicate such that* $\mathsf{exactly2}\ X$ *holds iff* $\mathsf{atleast2}\ X \wedge \neg\mathsf{atleast3}\ X$.

Definition 1.10. *We define* $\mathsf{exactly3}$ *to be a unary predicate such that* $\mathsf{exactly3}\ X$ *holds iff* $\mathsf{atleast3}\ X \wedge \neg\mathsf{atleast4}\ X$.

Definition 1.11. *We define* $\mathsf{exactly4}$ *to be a unary predicate such that* $\mathsf{exactly4}\ X$ *holds iff* $\mathsf{atleast4}\ X \wedge \neg\mathsf{atleast5}\ X$.

Definition 1.12. *We define* $\mathsf{exactly5}$ *to be a unary predicate such that* $\mathsf{exactly5}\ X$ *holds iff* $\mathsf{atleast5}\ X \wedge \neg\mathsf{atleast6}\ X$.

Definition 1.13. *We define* $\mathsf{exactly6}$ *to be a unary predicate such that* $\mathsf{exactly6}\ X$ *holds iff* $\mathsf{atleast6}\ X \wedge \neg\mathsf{atleast7}\ X$.

Axiom 1.1. $\forall XY.X \subseteq Y \Rightarrow Y \subseteq X \Rightarrow X = Y$.

Axiom 1.2. $\forall x.x \notin x$.

Axiom 1.3. $\forall xy.x \in y \Rightarrow y \notin x$.

Axiom 1.4. $\forall x.x \notin 0$.

Axiom 1.5. $\forall i.i \in 1 \Leftrightarrow i = 0$.

Axiom 1.6. $\forall i.i \in 2 \Leftrightarrow i = 0 \vee i = 1$.

Axiom 1.7. $\forall i.i \in 3 \Leftrightarrow i = 0 \vee i = 1 \vee i = 2$.

Axiom 1.8. $\forall i.i \in 4 \Leftrightarrow i = 0 \vee i = 1 \vee i = 2 \vee i = 3$.

Axiom 1.9. $\forall XY.Y \in \wp\ X \Leftrightarrow Y \subseteq X$.

Axiom 1.10. $\forall xy.y \in \{x\} \Leftrightarrow y = x$.

Axiom 1.11. $\forall XYz.z \in X \cup Y \Leftrightarrow z \in X \vee z \in Y$.

Axiom 1.12. $\forall XYz.z \in X \cap Y \Leftrightarrow z \in X \wedge z \in Y$.

Axiom 1.13. $\forall XYz.z \in X \setminus Y \Leftrightarrow z \in X \wedge z \notin Y$.

1.2 Discriminator and Shallow Rules

DISCRIMINATOR is a first-order automated theorem prover. Unlike most first-order automated theorem provers, e.g., Prover9 [9], E [11] and Vampire [8], DISCRIMINATOR does not use clause normalization, does not use resolution and does not use metavariables. Since there are no metavariables there is no need for unification (or even matching) to instantiate metavariables.

The search procedure DISCRIMINATOR uses proceeds in three phases, each of which is affected by a variety of parameters. The three phases are the opening phase, the search phase and the closing phase. During the opening phase DISCRIMINATOR may (optionally) perform operations such as decomposing logical operators, splitting the goal into multiple subgoals and expanding away some abbreviations. The opening phases then passes each subgoal (one at a time) to the main search phase (which we will simply call the *search phase*). The search phase performs a complete search (for first-order logic) based largely on the calculus of Satallax [6, 4, 5], although with no need for $\beta\eta$-normalization since we are in the first-order case. During search DISCRIMINATOR processes propositions and instantiations to create ground instances of rules and information about these instances are passed to the SAT solver MiniSat [7]. The closing phase is (optionally) activated if a certain number of abstract steps have been taken in the search. During the closing phase completeness is purposefully abandoned. The closing phase continues to process the propositions and instantiations generated during the search, but does not generate all the new propositions and instantiations required for completeness.

The main new feature of DISCRIMINATOR (not in Satallax) is the production and use of *shallow rules*. For the moment we consider only a special form of shallow rule: a *linear predicate shallow rule*. These will be given in the following format:

$$\Gamma|\psi \Rightarrow \varphi_1, \ldots, \varphi_n$$

Here Γ is a list of variables and ψ, φ_1, …, φ_n are formulas that may contain free variables from Γ. Here ψ is the *trigger formula* of the rule and will always be one of the following forms:

- $pt_1 \cdots t_m$ where each t_i is either a variable in Γ or of the form $fx_1 \cdots x_k$ where each x_j is a variable in Γ. Every variable in Γ must occur exactly once in $pt_1 \cdots t_m$. The predicate p may be equality.

- $\neg pt_1 \cdots t_m$ where each t_i is either a variable in Γ or of the form $fx_1 \cdots x_k$ where each x_j is a variable in Γ. Every variable in Γ must occur exactly once in $pt_1 \cdots t_m$. The predicate p may be equality.

When we are processing a formula Ψ one can easily determine if the trigger formula matches the formula and uniquely determine a substitution θ for all the variables in Γ such that $\theta(\psi) = \Psi$ without doing general first-order matching since all the variables occur (exactly once) in a shallow position of ψ. Using θ we have (potentially) new (ground) propositions $\theta(\varphi_1)$, ..., $\theta(\varphi_n)$. If these propositions are new, they will be added to the priority queue to be processed later. Additionally a propositional clause recording the relationship between Ψ and $\theta(\varphi_1)$, ..., $\theta(\varphi_n)$ is created and given to MiniSat.

Shallow rules can be seen as analogous to the rules for if-then-else and choice developed for Satallax [2]. In those cases the rules were sufficient to obtain completeness with no extra axioms for if-then-else or choice. In the case of Discriminator we generate shallow rules from propositions without reason to believe the shallow rule can completely replace the source proposition during the search. However, as a heuristic we typically assign propositions low priority if they produce at least one shallow rule.

The set theory axioms from the previous section produce many linear predicate shallow rules. We record these here.

Axiom 1.1 produces the following rules:

$$x, y | x \neq y \Rightarrow x \not\subseteq y, y \not\subseteq x \tag{1.1}$$

$$x, y | x \subseteq y \Rightarrow x = y, y \not\subseteq x \tag{1.2}$$

$$x, y | y \subseteq x \Rightarrow x = y, x \not\subseteq y \tag{1.3}$$

Axiom 1.2 produces the following rule:

$$x, y | x \in y \to y \neq x \tag{1.4}$$

The reader may have expected the rule to appear as $x | x \in x \to \cdot$. However, the x occurs twice in $x \in x$, so this would not be a linear shallow predicate rule.

Axiom 1.3 produces the following rule:

$$x, y | x \in y \Rightarrow y \notin x \tag{1.5}$$

Axiom 1.4 produces the following rule:

$$x | x \in 0 \Rightarrow \cdot \tag{1.6}$$

Axiom 1.5 produces the following rules:

$$x | x \in 1 \Rightarrow x = 0 \tag{1.7}$$

$$x | x = 0 \Rightarrow x \in 1 \tag{1.8}$$

$$x | x \notin 1 \Rightarrow x \neq 0 \tag{1.9}$$

$$x | x \neq 0 \Rightarrow x \notin 1 \tag{1.10}$$

Axiom 1.6 produces the following rules:

$$x | x \in 2 \Rightarrow x = 0 \vee x = 1 \tag{1.11}$$

$$x | x = 0 \Rightarrow x \in 2 \tag{1.12}$$

$$x|x = 1 \Rightarrow x \in 2 \tag{1.13}$$
$$x|x \notin 2 \Rightarrow \neg(x = 0 \vee x = 1) \tag{1.14}$$
$$x|x \neq 0 \Rightarrow x = 1, x \notin 2 \tag{1.15}$$
$$x|x \neq 1 \Rightarrow x = 0, x \notin 2 \tag{1.16}$$

There are 8 similar rules produced from Axiom 1.7 and 10 rules produced from Axiom 1.8. We show 3 of the 8 rules produced from Axiom 1.7.

$$x|x \in 3 \Rightarrow x = 0 \vee x = 1 \vee x = 2 \tag{1.17}$$
$$x|x \neq 1 \Rightarrow x = 0, x = 2, x \notin 3 \tag{1.18}$$
$$x|x \notin 3 \Rightarrow \neg(x = 0 \vee x = 1 \vee x = 2) \tag{1.19}$$

We show 3 of the 10 rules produced from Axiom 1.8.

$$x|x \in 4 \Rightarrow x = 0 \vee x = 1 \vee x = 2 \vee x = 3 \tag{1.20}$$
$$x|x = 1 \Rightarrow x \in 4 \tag{1.21}$$
$$x|x = 2 \Rightarrow x \in 4 \tag{1.22}$$

Axiom 1.9 generates four shallow rules:

$$x, y|y \in \wp x \Rightarrow y \subseteq x \tag{1.23}$$
$$x, y|y \notin \wp x \Rightarrow y \nsubseteq x \tag{1.24}$$
$$x, y|y \subseteq x \Rightarrow y \in \wp x \tag{1.25}$$
$$x, y|y \nsubseteq x \Rightarrow y \notin \wp x \tag{1.26}$$

Axiom 1.10 produces the following rules:

$$x, y|y \in \{x\} \Rightarrow y = x \tag{1.27}$$
$$x, y|y = x \Rightarrow y \in \{x\} \tag{1.28}$$
$$x, y|y \notin \{x\} \Rightarrow y \neq x \tag{1.29}$$
$$x, y|y \notin \{x\} \Rightarrow x \neq y \tag{1.30}$$
$$x, y|y \neq x \Rightarrow y \notin \{x\} \tag{1.31}$$

Axiom 1.11 produces the following rules:

$$x, y, z|z \in x \cup y \Rightarrow z \in x \vee z \in y \tag{1.32}$$
$$x, y, z|z \notin x \cup y \Rightarrow \neg(z \in x \vee z \in y) \tag{1.33}$$

Axiom 1.12 produces the following rules:

$$x, y, z|z \in x \cap y \Rightarrow z \in x \wedge z \in y \tag{1.34}$$
$$x, y, z|z \notin x \cap y \Rightarrow \neg(z \in x \wedge z \in y) \tag{1.35}$$

Axiom 1.13 produces the following rules:

$$x, y, z|z \in x \setminus y \Rightarrow z \in x \wedge z \notin y \tag{1.36}$$
$$x, y, z|z \notin x \setminus y \Rightarrow \neg(z \in x \wedge z \notin y) \tag{1.37}$$

Chapter 2

Theorem 1: 2 has Exactly 2 Elements

Theorem 1: $\mathsf{exactly2}\ 2$.

Let us first give a quick informal proof to give an idea what an automated theorem prover would need to do to prove this theorem from the axioms.

Proof. We need to prove two things: $\mathsf{atleast2}\ 2$ and $\neg\mathsf{atleast3}\ 2$. We first prove $\mathsf{atleast2}\ 2$. By Definition 1.3 we need to give some $Y \in 2$ such that $2 \not\subseteq \wp Y$. We take Y to be 0. We easily have $0 \in 2$ from Axiom 1.6. It remains to prove $2 \not\subseteq \wp 0$. Assume $2 \subseteq \wp 0$. We have $1 \in 2$ from Axiom 1.6 and so we must have $1 \in \wp 0$ (using Definition 1.1). By Axiom 1.9 we have $1 \subseteq 0$. We know $0 \in 1$ by Axiom 1.5 and so $0 \in 0$. The conclusion $0 \in 0$ contradicts both Axiom 1.2 and Axiom 1.4, providing two ways to complete this subproof.

Next we prove $\neg\mathsf{atleast3}\ 2$. By Definition 1.10 there must be some $Y \subseteq 2$ such that $2 \not\subseteq Y$ and $\mathsf{atleast2}\ Y$. By Definition 1.1 there must be some $a \in 2$ with $a \notin Y$. By Definition 1.9 there must be some $b \in Y$ with $Y \not\subseteq \wp b$. By Definition 1.1 there must be some $c \in Y$ with $c \notin \wp b$. Since $a \notin Y$, $b \in Y$ and $c \in Y$, we know $a \neq b$ and $a \neq c$. Using $c \notin \wp b$ we can also argue $b \neq c$ (using Axioms 1.9 and 1.1). Since $Y \subseteq 2$, we know $b \in 2$ and $c \in 2$. Applying Axiom 1.6 with each of a, b and c we have eight cases, each of which is in contradiction with $a \neq b$, $a \neq c$ and $b \neq c$. □

Let us now informally describe how DISCRIMINATOR searches for and finds some proofs of the theorem.

In each successful search DISCRIMINATOR uses the opening to recognize that certain predicates can be considered abbreviations, including $\subseteq$, $\mathsf{exactly2}$, $\mathsf{atleast2}$ and $\mathsf{atleast3}$. These are expanded during the opening and the subgoal is split into two subgoals, corresponding to proving $\mathsf{atleast2}\ 2$ and $\neg\mathsf{atleast3}\ 2$. In each proof below we describe the search for these two subgoals separately.

We write $s \hat{\subseteq} t$ as notation for $\forall z.z \in s \to z \in t$ and $s \hat{\not\subseteq} t$ for the negation.

2.1 Theorem 1 Proof 1

2.1.1 Search for Subgoal 1

The first subgoal asserts the axioms above (with definitional axioms removed and with abbreviations expanded in the remaining axioms) and the additional proposition:

$$\neg\exists Y \in 2.2 \hat{\not\subseteq} \wp Y$$

where we write $s \hat{\subseteq} t$ as notation for $\forall z.z \in s \to z \in t$ and $s \hat{\not\subseteq} t$ for the negation. (Recall that the predicate $\subseteq$ has been expanded away already.) These asserted propositions are analyzed to see which produce shallow rules. In this case *every* asserted proposition produces at least one shallow rule. We saw shallow rules for the axioms in Section 1.2 and these remain mostly the same. The exceptions are caused by the expansion of $\subseteq$ in Axioms 1.1 and 1.9. Due to this expansion the modified Axiom 1.1 yields only the shallow rule

$$x, y | x \neq y \Rightarrow x \hat{\not\subseteq} y, y \hat{\not\subseteq} x \tag{2.1}$$

and the modified Axiom 1.9 yields only the shallow rules

$$x, y | y \in \wp x \Rightarrow y \hat{\subseteq} x \tag{2.2}$$

$$x, y | y \notin \wp x \Rightarrow y \hat{\not\subseteq} x \tag{2.3}$$

A legend assigning numbers to the propositions considered during the search is given in Table 2.1. A figure showing the steps leading to the proof is given in Figure 2.1. For each asserted formula a unit clause is sent to MiniSat. For example, $\boxed{14}$ represents the proposition

$$\exists Y \in 2.2 \hat{\not\subseteq} \wp Y$$

and the unit clause $\boxed{\text{-14}}$ (representing the negation of the conclusion) is sent to MiniSat. For $i \in \{1, \ldots, 13\}$, $\boxed{\text{i}}$ represents one of the axioms (with $\subseteq$ expanded) and the unit clause $\boxed{\text{i}}$ is sent to MiniSat. The ones that play a role are $\boxed{5}$, $\boxed{7}$, $\boxed{8}$, $\boxed{11}$, $\boxed{12}$ and $\boxed{13}$ (see Table 2.1).

The negated conclusion

$$\neg\exists Y \in 2.2 \hat{\not\subseteq} \wp Y$$

yields two shallow rules:

$$x | x \in 2 \Rightarrow 2 \hat{\subseteq} \wp x \tag{2.4}$$

$$x, y | x \notin \wp y \Rightarrow x \notin 2, y \notin 2 \tag{2.5}$$

Due to Shallow Rule 2.5, whenever we process a proposition of the form $s \notin \wp t$ we will generate the propositions $s \notin 2$ and $t \notin 2$ along with the propositional information relating these three propositions.

The story of how DISCRIMINATOR proceeds to search for this subgoal is somewhat roundabout.[1]

[1] This is a polite way to refer to what appears to be a drunken stumbling.

$\boxed{5}$	$\forall x.x \in 4 \leftrightarrow x = 0 \vee x = 1 \vee x = 2 \vee x = 3$	$\boxed{7}$	$\forall i.i \in 2 \Leftrightarrow i = 0 \vee i = 1$
$\boxed{8}$	$\forall i.i \in 1 \Leftrightarrow i = 0$	$\boxed{11}$	$\forall x.x \notin x$
$\boxed{12}$	$\forall XY.X \hat{\subseteq} Y \Rightarrow Y \hat{\subseteq} X \Rightarrow X = Y$	$\boxed{13}$	$\forall XY.Y \in \wp\, X \Leftrightarrow Y \hat{\subseteq} X$
$\boxed{14}$	$\exists Y \in 2.2 \hat{\not\subseteq} \wp Y$	$\boxed{17}$	$(1 \in 4 \leftrightarrow 1 = 0 \vee 1 = 1 \vee 1 = 2 \vee 1 = 3)$
$\boxed{20}$	$1 = 0 \vee 1 = 1 \vee 1 = 2 \vee 1 = 3$	$\boxed{33}$	$1 = 0 \vee 1 = 1 \vee 1 = 2$
$\boxed{41}$	$1 = 0 \vee 1 = 1$	$\boxed{50}$	$1 = 1$
$\boxed{51}$	$1 = 0$	$\boxed{54}$	$1 \in 2$
$\boxed{58}$	$2 \hat{\subseteq} \wp 1$	$\boxed{60}$	$0 \in 2 \rightarrow 0 \in \wp 1$
$\boxed{63}$	$2 \in 2 \rightarrow 2 \in \wp 1$	$\boxed{65}$	$0 \in \wp 1$
$\boxed{66}$	$0 \in 2$	$\boxed{69}$	$2 \in \wp 1$
$\boxed{74}$	$2 \hat{\subseteq} 1$	$\boxed{76}$	$0 \in 2 \rightarrow 0 \in 1$
$\boxed{77}$	$1 \in 2 \rightarrow 1 \in 1$	$\boxed{81}$	$0 \in 1$
$\boxed{85}$	$0 = 1$	$\boxed{86}$	$0 = 0$
$\boxed{87}$	$0 = 2$	$\boxed{89}$	$1 \hat{\subseteq} 0$
$\boxed{90}$	$0 \hat{\subseteq} 1$	$\boxed{97}$	$0 \in 4$
$\boxed{103}$	$0 = 0 \vee 0 = 1 \vee 0 = 2 \vee 0 = 3$	$\boxed{104}$	$0 = 0 \vee 0 = 1 \vee 0 = 2$
$\boxed{105}$	$0 = 0 \vee 0 = 1$	$\boxed{108}$	$2 \hat{\subseteq} \wp 0$
$\boxed{113}$	$2 \in 2 \rightarrow 2 \in \wp 0$	$\boxed{115}$	$1 \in \wp 0$
$\boxed{116}$	$2 \in \wp 0$	$\boxed{121}$	$2 \hat{\subseteq} 0$
$\boxed{128}$	$1 \in 0$		

Table 2.1: Propositions in Subgoal 1 Search

When a proposition produces at least one shallow rule we often (heuristically) give a lower priority to processing the source proposition. In this case since all the propositions produce shallow rules, all the initial propositions have a low priority. For this reason the search will begin by processing instantiations.

The complete calculus for Discriminator only requires instantiating with terms that occur on the left or right hand side of a disequation (*discriminating terms*) or with a default term if no term is discriminating. In this case there are no disequations (yet) so no term is discriminating. One option in Discriminator is to seed the initial set of instantiations with all (ground) subterms of the problem. The only ground subterms of the propositions asserted in this subgoal are 0, 1, 2, 3 and 4. There are no discriminating terms in the subgoal, so these are the only instantiations in the beginning.

Processing an instantiation before processing any positive universally quantified propositions or negative existentially quantified propositions only has a bookkeeping effects. The search begins by processing these five instantiations and then looking for one of the (low priority) propositions to process. Steps 0 (processing instantiation 2), 2 (processing instantation 1) and 3 (processing instantiation 0) are the first steps shown in Figure 2.1. No rules apply and no propositions, instantiations or clauses are generated. The side effect (not shown in Figure 2.1) is that these instantiations are now available when future propositions are processed.

The first proposition processed is Axiom 1.8 ($\boxed{5}$ in Table 2.1 and Figure 2.1). This is instantiated with each of 0, 1, 2, 3 and 4. The instance that will play a role in the

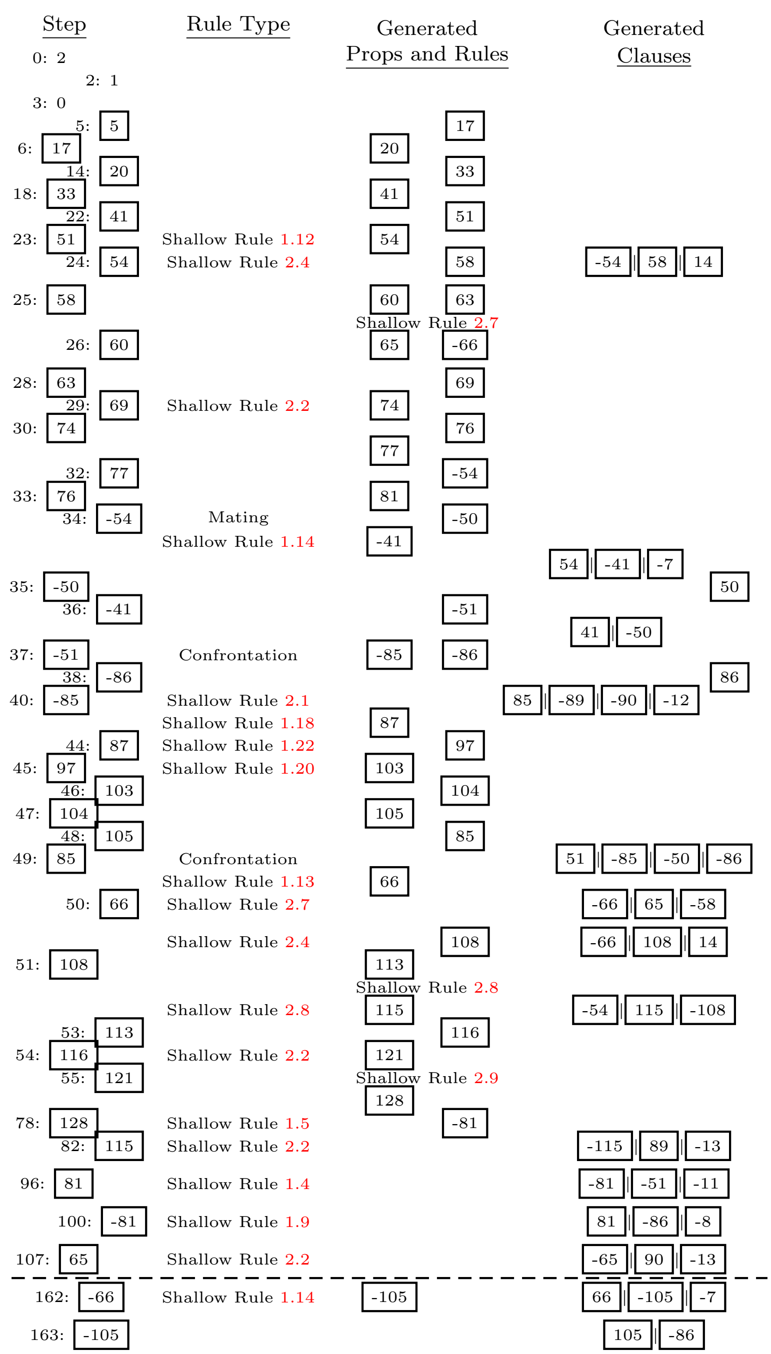

Figure 2.1: Search Steps Leading to Proof of Subgoal 1

successful proof is the one given by 1:

$$\boxed{17} : 1 \in 4 \leftrightarrow 1 = 0 \vee 1 = 1 \vee 1 = 2 \vee 1 = 3.$$

This may seem surprising since this seems to have nothing directly to do with proving 2 has at least 2 elements. However, processing the proposition and subsequent subformulas leads to processing the propositions $1 = 0$ and $1 = 1$. Since the proposition above is an equivalence, $1 = 0$ and $1 = 1$ occur both positively and negatively as subformulas. When processing $1 = 0$ (a positive proposition), the proposition $1 \in 2$ is generated due to Shallow Rule 1.12. This illustrates the somewhat unexpected way propositions are sometimes generated. If 1 were equal to 0, then 1 would be in 2 so we consider $1 \in 2$. Of course, $1 \in 2$ is both true and a reasonable proposition to consider, even though it was generated by considering the false proposition $1 = 0$. The reader should keep in mind that "processing" a proposition has nothing to do with whether the proposition is true or false, but only to do with determining its relationship to other propositions.

When processing $1 \in 2$, Shallow Rule 2.4 leads DISCRIMINATOR to generate $2 \hat{\subseteq} \wp 1$. Again, this is a somewhat unexpected way to obtain $2 \hat{\subseteq} \wp 1$ since it is true, but DISCRIMINATOR is using the negation of atleast2 2 to obtain it. In Table 2.1 and Figure 2.1, $1 \in 2$ is $\boxed{54}$ and $2 \hat{\subseteq} \wp 1$ is $\boxed{58}$. Shallow Rule 2.4 also yields the propositional clause $\boxed{-54} \| \boxed{58} \| \boxed{14}$ to send to MiniSat. (This is the first clause generated during the search that will play a role in the unsatisfiability of the clauses sent to MiniSat.) Recall that $\boxed{-14}$ is a unit clause from the initialization of the search.

Recall that $2 \hat{\subseteq} \wp 1$ is notation for $\forall z.z \in 2 \rightarrow z \in \wp 1$. When DISCRIMINATOR processes this universally quantified formula it generates the new shallow rules

$$y, z | z \notin \wp y \Rightarrow y \neq 1, z \notin 2 \tag{2.6}$$

and

$$z | z \in 2 \Rightarrow z \in \wp 1 \tag{2.7}$$

Only this second rule will play a role in the eventual proof. DISCRIMINATOR also instantiates the formula with all the five instantations 0, 1, 2, 3 and 4. Two instances play a role in the proof. The instance with 0 (i.e., $0 \in 2 \rightarrow 0 \in \wp 1$, abbreviated as $\boxed{60}$) plays a direct role in the proof. The instance with 2 (i.e., $2 \in 2 \rightarrow 2 \in \wp 1$, abbreviated as $\boxed{63}$) plays an indirect, but still important, role during the search. Processing the instance with 0 (in Step 26) generates propositions $0 \notin 2$ and $0 \in \wp 1$, and both will participate in the proof. Processing the instance with 2 generates propositions $2 \notin 2$ and $2 \in \wp 1$, but only $2 \in \wp 1$ will participate in the proof.

Step 29 in Figure 2.1 corresponds to processing $2 \in \wp 1$. Shallow Rule 2.2 generates the proposition $2 \hat{\subseteq} 1$. Processing (in Step 30) this yields $0 \in 2 \rightarrow 0 \in 1$ ($\boxed{76}$) and $1 \in 2 \rightarrow 1 \in 1$ ($\boxed{77}$) leading DISCRIMINATOR to generate propositions $0 \in 1$ and $1 \notin 2$. (Recall above DISCRIMINATOR had already processed the positive proposition $1 \in 2$, see Step 24 in Figure 2.1.) DISCRIMINATOR processes $1 \notin 2$ in Step 34, delaying processing $0 \in 1$ until Step 96.

Two important actions happen when processing $1 \notin 2$. Since $1 \in 2$ was processed before, it is mated with $1 \notin 2$[2] producing disequations $1 \neq 1$ and $2 \neq 2$. In addition Shallow Rule 1.14 gives that if $1 \notin 2$, then $\neg(1 = 0 \vee 1 = 1)$. This results in generating the proposition $\neg(1 = 0 \vee 1 = 1)$ ($\boxed{-41}$) and the clause $\boxed{54}\|\boxed{-41}\|\boxed{-7}$ sent to MiniSat. Since $\boxed{7}$ is a unit clause from the initialization, we must have either $1 \in 2$ or $\neg(1 = 0 \vee 1 = 1)$.

Processing $1 \neq 1$ yields the unit clause $\boxed{50}$ representing the fact that $1 = 1$ (by an equality rule of the calculus). Processing $\neg(1 = 0 \vee 1 = 1)$ yields an obvious relationship with $1 = 1$ represented by the clause $\boxed{41}\|\boxed{-50}$. Essentially at this point we know $1 = 0 \vee 1 = 1$ holds, and so $1 \in 2$ holds. Using the information represented by the clause $\boxed{-54}\|\boxed{58}\|\boxed{14}$ we can further infer that $2 \hat{\subseteq} \wp 1$ must hold.

When the proposition $\neg(1 = 0 \vee 1 = 1)$ was processed, the proposition $1 \neq 0$ was generated. In Step 37, $1 \neq 0$ is processed. Recall that the positive proposition $1 = 0$ was processed above. When $1 \neq 0$ is processed the disequation $1 = 0$ is *confronted* by the equation $1 = 0$[3] yielding as a side effect the disequations $0 \neq 1$ and $0 \neq 0$. Processing $0 \neq 0$ (in Step 38) produces the unit clause $\boxed{86}$ form MiniSat, representing the truth of $0 = 0$.

When $0 \neq 1$ is processed (in Step 40) two relevant independent actions occur. Shallow Rule 2.1 yields the clause $\boxed{85}\|\boxed{-89}\|\boxed{-90}\|\boxed{-12}$ recording that if $0 \neq 1$, then either $0 \hat{\not\subseteq} 1$ or $1 \hat{\not\subseteq} 1$. Shallow Rule 1.18 generates the proposition $0 = 2$. When $0 = 2$ is processed Shallow Rule 1.22 generates the proposition $0 \in 4$. When $0 \in 4$ is processed the proposition $0 = 0 \vee 0 = 1 \vee 0 = 2 \vee 0 = 3$ is generated. By processing it and its subformulas eventually $0 = 1$ is processed.

Two important actions occur when $0 = 1$ is processed in Step 49. First $0 = 1$ confronts $1 \neq 0$ (processed earlier) to record the propositional information that both cannot be true. This is represented by the clause $\boxed{51}\|\boxed{-85}\|\boxed{-50}\|\boxed{-86}$. Second Shallow Rule 1.13 is triggered and generates the propositional formula $0 \in 2$.

When $0 \in 2$ is processed in Step 50 two shallow rules apply. Shallow Rule 2.7 produces the clause $\boxed{-66}\|\boxed{65}\|\boxed{-58}$, essentially representing that if $0 \in 2$, then $0 \in \wp 1$. Shallow Rule 2.4 produces the new proposition $2 \hat{\subseteq} \wp 0$ and the clause $\boxed{-66}\|\boxed{108}\|\boxed{14}$. Since $\boxed{-14}$ is a unit clause from the initialization, this clause means that if $0 \in 2$, then $2 \hat{\subseteq} \wp 0$.

Processing $2 \hat{\subseteq} \wp 0$ in Step 51 yields the proposition $2 \in 2 \to 2 \in \wp 0$ ($\boxed{113}$) by

[2] This is weird. It's almost certainly unnecessary to mate a proposition with its literal complement. In this case it happens to help. Since writing this description I changed the code to prevent mating formulas with their signed counterpart and prevent confrontations with their signed counterparts, with parameters to allow such "self matings" and "self confrontations." The proof described here depends on allowing self matings and self confrontations, but since changing the code alternative proofs have (allegedly) been found that do not require self matings or self confrontations. I need to look into these more closely to make sure there is nothing suspicious about the new proofs.

[3] Confrontation is essentially the Mating rule for equality, making use of symmetry of equality. See [2].

instantiating with 2. In addition the following new shallow rule is created:

$$x|x \in 2 \Rightarrow x \in \wp 0 \tag{2.8}$$

After creating this new shallow rule Discriminator checks if it is triggered by any previously processed propositions. In fact $1 \in 2$ (processed in Step 24) triggers the rule leading to the new proposition $1 \in \wp 0$ ($\boxed{115}$) and the clause $\boxed{-54}\|\boxed{115}\|\boxed{-108}$.

Processing $2 \in 2 \to 2 \in \wp 0$ leads to processing $2 \in \wp 0$. Shallow Rule 2.2 generates $2\hat{\subseteq}0$. Processing $2\hat{\subseteq}0$ creates the following new shallow rule:

$$x|x \in 2 \Rightarrow x \in 0 \tag{2.9}$$

This new shallow rule is triggered by $1 \in 2$ generating the proposition $1 \in 0$. Processing $1 \in 0$ (in Step 78) generates the proposition $0 \notin 1$ by Shallow Rule 1.5.

In Step 82 $1 \in \wp 0$ is processed. Shallow rule 2.2 yields clause $\boxed{-115}\|\boxed{89}\|\boxed{-13}$ representing that if $1 \in \wp 0$, then $1\hat{\subseteq}0$ (since $\boxed{13}$ is a unit clause from the initial assumptions).

In Step 96 $0 \in 1$ (generated in Step 33) is finally processed. Shallow Rule 1.4 yields the clause $\boxed{-81}\|\boxed{-51}\|\boxed{-11}$ meaning that if $0 \in 1$, then $1 \neq 0$.

In Step 100 $0 \notin 1$ (generated in Step 78) is processed. Shallow Rule 1.9 yields the clause $\boxed{81}\|\boxed{-86}\|\boxed{-8}$ essentially giving that if $0 \notin 1$, then $0 \neq 0$. Since we already know $0 = 0$ from Step 38, we now know $0 \in 1$. Combining this with the information from Step 96, we know $1 \neq 0$.

In Step 107 $0 \in \wp 1$ (generated in Step 26) is processed. Shallow Rule 2.2 yields the clause $\boxed{-65}\|\boxed{90}\|\boxed{-13}$, representing that if $0 \in \wp 1$, then $0\hat{\subseteq}1$.

Discriminator was called with a middle abstract time limit of 128. As a consequence Discriminator continues searching as usual by processing propositions for 20 more steps. After these steps Discriminator enters the closing phase. This is represented by the dashed line in Figure 2.1. In the closing phase, Discriminator continues to process propositions (and instantiations, if there are any) on the priority queue but no longer generates all new propositions and adds them to the priority queue for later processing. (This is, of course, very incomplete.) Depending on parameter settings Discriminator might not add any new propositions to the priority queue. For this search a parameter is set so that Discriminator will add a new proposition to the priority queue only if it is an implication, a conjunction, a disjunction or an equivalence (or the negation of one of these).

During the closing phase only two steps will be required to complete the proof. In Step 162 the formula $0 \notin 2$ (generated in Step 26) is processed. Shallow Rule 1.14 yields the proposition $\neg(0 = 0 \vee 0 = 1)$ and the clause $\boxed{66}\|\boxed{-105}\|\boxed{-7}$, representing that if $0 \notin 2$, then $\neg(0 = 0 \vee 0 = 1)$. Since $\neg(0 = 0 \vee 0 = 1)$ is the negation of a disjunction, it is added to the priority queue.

In the final step, Step 163, $\neg(0 = 0 \vee 0 = 1)$ is processed. The clause $\boxed{105}\|\boxed{-86}$ is sent to MiniSat and MiniSat reports unsatisfiability, completing the proof of the subgoal.

Before ending the discussion of this subgoal, let us reconsider the argument by using the MiniSat clauses to indicate how a refutation is reached. In the beginning there are

14 unit clauses: $\boxed{-14}$ and $\boxed{i}$ for $i \in \{1, \ldots, 13\}$. Two more unit clauses are produced during the search: 50 (for $1 = 1$) and 86 (for $0 = 0$). These unit clauses can be propagated (leading directly to unsatisfiability) as follows. The clause $\boxed{41}\boxed{-50}$ gives the unit $\boxed{41}$ (for $1 = 0 \vee 1 = 1$). The clause $\boxed{54}\boxed{-41}\boxed{-7}$ gives $\boxed{54}$ (for $1 \in 2$). The clause $\boxed{-54}\boxed{58}\boxed{14}$ gives $\boxed{58}$ (for $2\hat{\subseteq}\wp 1$).

The clause $\boxed{81}\boxed{-86}\boxed{-8}$ gives the unit $\boxed{81}$ (for $0 \in 1$) and the clause $\boxed{105}\boxed{-86}$ gives the unit $\boxed{105}$ (for $0 = 0 \vee 0 = 1$). Now $\boxed{66}\boxed{-105}\boxed{-7}$ gives the unit $\boxed{66}$ (for $0 \in 2$). Now $\boxed{-66}\boxed{65}\boxed{-58}$ gives the unit $\boxed{65}$ (for $0 \in \wp 1$) and $\boxed{-66}\boxed{108}\boxed{14}$ gives the unit $\boxed{108}$ (for $2\hat{\subseteq}\wp 0$). Note that $2\hat{\subseteq}\wp 0$ is actually false, but it is implied by the negation of the conclusion we with to prove (represented by $\boxed{-14}$).

At this point $\boxed{-54}\boxed{115}\boxed{-108}$ gives the unit $\boxed{115}$ (for $1 \in \wp 0$). Now $\boxed{-115}\boxed{89}\boxed{-13}$ gives the unit $\boxed{89}$ (for $1\hat{\subseteq}0$). Additionally $\boxed{-65}\boxed{90}\boxed{-13}$ gives the unit $\boxed{90}$ (for $0\hat{\subseteq}1$). These last two units combine with $\boxed{85}\boxed{-89}\boxed{-90}\boxed{-12}$ to give $\boxed{85}$ (for $0 = 1$). Now $\boxed{51}\boxed{-85}\boxed{-50}\boxed{-86}$ gives $\boxed{51}$ (for $1 = 0$).

We are now in conflict with the clause $\boxed{-81}\boxed{-51}\boxed{-11}$. That is, we have derived $0 \in 1$ and $1 = 0$ above, but also derived that both of these cannot be true.

2.1.2 Search for Subgoal 2

When considering the second subgoal the opening phase can break the problem down further after expanding atleast3, atleast2 and $\subseteq$. As a consequence the opening creates four fresh eigenvariables Y, a, b and c. In addition to the axioms (with abbreviations expanded), the search includes the following propositions:

1. $Y\hat{\subseteq}2$
2. $a \in 2$,
3. $b \in Y$,
4. $c \in Y$,
5. $c \notin \wp b$ and
6. $a \notin Y$.

The proposition $Y\hat{\subseteq}2$ produces the following shallow rule:

$$x|x \in Y \Rightarrow x \in 2 \tag{2.10}$$

The other propositions above do not produce a shallow rule and so they are given priority when the search begins. Other shallow rules that are produced from the axioms (with $\subseteq$ expanded) and are used in the search are Shallow Rule 1.11 and 2.2. The important propositions used in the search are given in Table 2.2 and the steps leading to a proof are shown in Figure 2.2. MiniSat is initially given unit clauses corresponding

Step	Rule Type	Generated Props	Generated Clauses
1: 15	Shallow Rule 2.10	22	-15 ‖ 22 ‖ -12
2: 22	Shallow Rule 1.11	25	-22 ‖ 25 ‖ -7
5: 25			-25 ‖ 33 ‖ 32
6: -18	Shallow Rule 2.1	-37	18 ‖ -37 ‖ -17
11: -19	Mating	-45 -47	-15 ‖ 19 ‖ -47 ‖ -45
12: -47			47
27: -37		-95	37 ‖ -95
28: -95		96 -97	95 ‖ 96 95 ‖ -97
29: 96	Mating	-101	
33: -101			
39: 14	Mating	-133	-14 ‖ 19 ‖ -47 ‖ -133
	Shallow Rule 2.10	139	-14 ‖ 139 ‖ -12
40: -133			
45: 139	Shallow Rule 1.11	165	-139 ‖ 165 ‖ -7
46: 165		142 146	-165 ‖ 142 ‖ 146
47: 142	Confrontation	-167	133 ‖ -142 ‖ -167 ‖ -141
	Confrontation	-168	
48: -167			167
58: 146	Confrontation		133 ‖ -146 ‖ -167 ‖ -145
60: 13	Mating	-194	
	Shallow Rule 1.11	199	-13 ‖ 199 ‖ -7
61: -194			194
62: 199		141 145	-199 ‖ 141 ‖ 145
63: 145			
71: 141			
75: -97	Mating	-226	-96 ‖ 97 ‖ -168 ‖ -226
78: -226			226
164: -45	Confrontation		45 ‖ -141 ‖ -33 ‖ -194 45 ‖ -145 ‖ -32 ‖ -194
176: -168	Confrontation		168 ‖ -146 ‖ -32 ‖ -167 168 ‖ -142 ‖ -33 ‖ -167

Figure 2.2: Search Steps Leading to Proof of Subgoal 2

7	$\forall i.i \in 2 \Leftrightarrow i = 0 \lor i = 1$	12	$Y \hat{\subseteq} 2$
13	$a \in 2$	14	$b \in Y$
15	$c \in Y$	17	$\forall XY.Y \in \wp\, X \Leftrightarrow Y \hat{\subseteq} X$
18	$c \in \wp b$	19	$a \in Y$
22	$c \in 2$	25	$c = 0 \lor c = 1$
32	$c = 1$	33	$c = 0$
37	$c \hat{\subseteq} b$	45	$c = a$
47	$Y = Y$	77	$1 = 1$
95	$d \in c \to d \in b$	96	$d \in c$
97	$d \in b$	101	$d = c$
133	$b = a$	139	$b \in 2$
141	$a = 0$	142	$b = 0$
145	$a = 1$	146	$b = 1$
165	$b = 0 \lor b = 1$	167	$b = b$
168	$c = b$	194	$a = a$
199	$a = 0 \lor a = 1$	226	$d = d$

Table 2.2: Propositions in Subgoal 2 Search

to the axioms and assumptions of the subgoal. These unit clauses are 17 positive unit clauses $\boxed{i}$ for $i \in \{1, \ldots, 17\}$ and the two negative unit clauses $\boxed{-18}$ and $\boxed{-19}$.

In Step 1 $c \in Y$ is processed. Shallow Rule 2.10 generates the formula $c \in 2$ and a clause that essentially tells us that $c \in 2$ is true. In Step 2 $c \in 2$ is processed. Shallow Rule 1.11 generates the formula $c = 0 \lor c = 1$ and a clause that essentially tells us $c = 0 \lor c = 1$ is true. In Step 5 $c = 0 \lor c = 1$ is processed. The disjunction rule gives us the formulas $c = 0$ and $c = 1$ and a propositional clause telling us one of these must be true. Although the formulas $c = 0$ and $c = 1$ are generated, we do not list them as generated in Figure 2.2 since they will not need to be processed in order to obtain a proof. (They are both processed during the search, but this does not contribute to the ultimate success.)

In Step 6 $c \notin \wp b$ is processed. Shallow Rule 2.2 gives the formula $c \hat{\nsubseteq} b$ and a clause that essentially means $c \hat{\nsubseteq} b$ is true. We return to process in $c \hat{\nsubseteq} b$ at Step 27.

In Step 11 $a \notin Y$ is processed and is mated with $c \in Y$. This generates two disequations $c \neq a$ and $Y \neq Y$, as well as a clause essentially indicating that either $c \neq a$ or $Y \neq Y$ must be true. In Step 12 $Y \neq Y$ is processed resulting in a unit clause corresponding to the truth of $Y = Y$. Hence we now know $c \neq a$.

In Step 27 $c \hat{\nsubseteq} b$ is processed. Recall that this proposition is actually

$$\neg(\forall x.x \in c \to x \in b).$$

When Discriminator processes an existentially quantified proposition or a negated universally quantified proposition, it creates a fresh eigenvariable as a witness. Let us call this fresh eigenvariable d. The new proposition $\neg(d \in c \to d \in b)$ is generated as well as a clause that essentially says $d \in c \to d \in b$ is false. Note that this is different to the way such quantifiers are treated by most first-order automated theorem provers. Most first-order automated theorem provers eliminate these quantifiers using Skolem functions in a preprocessing step. The treatment of such quantifiers by DISCRIMINATOR is even different from other automated theorem provers that do not (always) use Skolem functions. For example TPS [1] (optionally) uses *selected variables* and, in order to maintain soundness, ensures acyclicity of an ordering relating selected variables and existential variables (see [10] for more information). DISCRIMINATOR does not use existential variables and so there is no need to maintain such an ordering.

In Step 28 $\neg(d \in c \to d \in b)$ is processed generating the propositions $d \in c$ and $d \notin b$ along with clauses recording that $d \in c$ is true and $d \notin b$ is true. In Step 29 $d \in c$ is processed and mated with $c \in \wp b$ generating the disequation $d \neq c$. In Step 33 $d \neq c$ is processed making it available for future confrontations.

In Step 39 $b \in Y$ is processed resulting in two important actions. First $b \in Y$ is mated with $a \notin Y$ resulting in the disequation $b \neq a$ and a clause that allows us to conclude that $b \neq a$ is true. Second the Shallow Rule 2.10 gives us the proposition $b \in 2$ and a clause allowing us to conclude $b \in 2$ is true. In Step 40 $b \neq a$ is processed, but this only has the bookkeeping effect of making $b \neq a$ available for future confrontations with positive equations. In Step 45 $b \in 2$ is processed and Shallow Rule 1.11 generates the disjunction $b = 0 \vee b = 1$ and a clause allowing us to conclude this disjunction is true. In Step 46 the disjunction is processed giving us equations $b = 0$ and $b = 1$ and a clause meaning one of these equations must be true.

In Step 47 the equation $b = 0$ is processed and confronts two previously processed disequations: $b \neq a$ and $d \neq c$. The confrontation with $b \neq a$ produces the disequation $b \neq b$ and a clause telling us that if $b = 0$ is true, then $a = 0$ is false or $b = b$ is false. The confrontation with $d \neq c$ produces the disequation $c \neq b$ which will be processed in the final step of the proof. In Step 48 $b \neq b$ is processed giving a unit clause corresponding to the truth of $b = b$. Now we know that if $b = 0$ is true, then $a = 0$ is false.

In Step 58 the equation $b = 1$ is processed and confronts $b \neq a$. This generates a clause that essentially says that if $b = 1$, then $a \neq 1$.

In Step 60 $a \in 2$ is processed giving two relevant results. Mating with $a \notin Y$ (processed in Step 11) generates the new disequation $a \neq a$. Shallow Rule 1.11 generates the disjunction $a = 0 \vee a = 1$ and a clause saying the disjunction is true. Step 61 processes $a \neq a$ producing the unit clause saying $a = a$ is true. Step 62 processes $a = 0 \vee a = 1$ producing the propositions $a = 0$ and $a = 1$ and producing a clause saying one of these propositions must be true. Steps 63 and 71 processes these two equations making them available for later confrontations.

Step 75 processes $d \notin b$ (from Step 28). Mating $d \notin b$ with $d \in c$ produces the disequation $d \neq d$ and a clause saying that either $c \neq b$ or $d \neq d$ must be true. Step 78 processes $d \neq d$ giving the unit clause saying $d = d$ is true. At this point we know $c \neq b$ must be true.

DISCRIMINATOR continues searching as above until 128 steps have been reached. As with the first subgoal at Step 128 the closing phase begins. After this point the only new propositions that will be added to the priority queue for future processing are implications, conjunctions, disjunctions or equivalences (or negations of one of these). During the closing phase two relevant steps will complete the proof.

Before describing these final two steps, let us summarize what we know so far. After Step 5 we know either $c = 0$ or $c = 1$. After Step 46 we know either $b = 0$ or $b = 1$. After Step 48 we know if $b = 0$, then $a \neq 0$. After Step 58 we know if $b = 1$, then $a \neq 1$. After Step 62 we know either $a = 0$ or $a = 1$. The only remaining possibilities to be ruled are when c has the same value as either a or b. We know $c \neq a$ from Step 12 and $c \neq b$ from Step 78, but the propositional clauses so far are still satisfiable.

In Step 164 $c \neq a$ is processed and is confronted by $a = 0$ and $a = 1$. The confrontation with $a = 0$ yields a clause meaning that if $c = 0$, then $a \neq 0$. The confrontation with $a = 1$ yields a clause meaning that if $c = 1$, then $a \neq 1$.

In the final step, Step 176, $c \neq b$ is processed and is confronted by $b = 0$ and $b = 1$. The confrontation with $b = 0$ yields a clause meaning that if $c = 0$, then $b \neq 0$. The confrontation with $b = 1$ yields a clause meaning that if $c = 1$, then $b \neq 1$. The set of propositional clauses is now unsatisfiable, completing the proof.

2.2 A Proof using a Cut

Discriminator can optionally be instructed to consider using a cut formula to split a subgoal into two subgoals (one in which the cut formula is assumed and one in which the cut formula is proven). In fact, Discriminator can be instructed to considering using up to n cut formulas (in principle splitting a subgoal into up to 2^n subgoals). If Discriminator is instructed to consider cut formulas, a maximum number of cut formulas and a random salt (a number) must be given. When the opening phase has reached a subgoal that could be sent to the search phase, it uses the salt to pseudo-randomly decide how many (if any) cut formulas are to be used. If $n > 0$ cut formulas will be used, Discriminator collects the ground subterms of the current subgoal and the binary and unary predicates in the subgoal. The set of candidate cut formulas are $s = t$ where s, t are distinct ground subterms, $p\ s$ where p is a unary predicate and $r\ s\ t$ where s, t are ground subterms and r is a binary relation.

In the case of the theorem $\mathsf{exactly2}\ 2$ Discriminator can easily find a proof if allowed to use (up to) 1 cut formula and given the salt 9. With these settings, the first subgoal will be split into two using the cut formula $0 \in 2$ and the second subgoal will not use a cut. We limit the main search phase to 64 abstract steps before the closing phase begins.

We describe the proof search below.

2.2.1 Subgoal 1

The first subgoal (proving 2 has at least two elements) is split into two subgoals using the cut formula $0 \in 2$. We describe the two searches independently below.

Due to expanding $\subseteq$, the relevant shallow rules produced by Axiom 1.9 have the forms:

$$x, y | y \in \wp x \Rightarrow y \hat{\subseteq} x \tag{2.11}$$

$\boxed{14}$	$0 \in 2$	$\boxed{18}$	$2\hat{\subseteq}\wp 0$
$\boxed{19}$	$0 = 0 \vee 0 = 1$	$\boxed{20}$	$0 = 1$
$\boxed{21}$	$0 = 0$	$\boxed{23}$	$2 \in 2 \to 2 \in \wp 0$
$\boxed{24}$	$0 \in \wp 0$	$\boxed{26}$	$\wp 0 \in 0$
$\boxed{27}$	$0\hat{\subseteq}0$	$\boxed{30}$	$2 \in \wp 0$
$\boxed{31}$	$2 \in 2$	$\boxed{36}$	$2\hat{\subseteq}0$
$\boxed{38}$	$0 \in 0$	$\boxed{43}$	$2 = 0 \vee 2 = 1$
$\boxed{44}$	$2 = 1$	$\boxed{46}$	$1 = 1$
$\boxed{52}$	$1 \in 0 \to 1 \in 0$	$\boxed{53}$	$1 \in 2 \to 1 \in \wp 0$
$\boxed{54}$	$1 \in 0$	$\boxed{57}$	$0 \in 1$
$\boxed{58}$	$1 \in \wp 0$	$\boxed{59}$	$1 \in 2$
$\boxed{62}$	$1\hat{\subseteq}0$	$\boxed{79}$	$1 = 0 \vee 1 = 1$

Table 2.3: Propositions in Subgoal 1(a) of Theorem 1 Search

$$x, y | y \notin \wp x \Rightarrow y \hat{\not\subseteq} x \tag{2.12}$$

Subgoal 1a (Assuming Cut Formula)

In this subgoal we assume the axioms as usual (with $\subseteq$ expanded), the cut formula $0 \in 2$ and the negation of the conclusion:

$$\exists Y \in 2.2 \hat{\not\subseteq} \wp Y.$$

Before the search begins MiniSat is given unit clauses corresponding to the assumptions of the subgoal. These are $\boxed{i}$ for $i \in \{1, \ldots, 14\}$ and $\boxed{-15}$ (corresponding to the negation of the conclusion). Again the negated conclusion

$$\neg \exists Y \in 2.2 \hat{\not\subseteq} \wp Y$$

yields Shallow Rules 2.13 and 2.14.

$$x | x \in 2 \Rightarrow 2 \hat{\subseteq} \wp x \tag{2.13}$$

$$x, y | x \notin \wp y \Rightarrow x \notin 2, y \notin 2 \tag{2.14}$$

The (nonaxiom) formulas used in the proof are in Table 2.3 and the relevant steps in the proof are in Figure 2.3.

Step 0 processes the instantiation 2 making it available for use later.

Step 1 processes $0 \in 2$ triggering Shallow Rule 2.13 producing the proposition $2\hat{\subseteq}\wp 0$ and a clause indicating $2\hat{\subseteq}\wp 0$ is true and triggering Shallow Rule 1.11 producing the proposition $0 = 0 \vee 0 = 1$. Step 2 processes $0 = 0 \vee 0 = 1$ producing the proposition $0 = 1$. Step 3 processes $0 = 1$ making it available for later confrontations (see Step 26).

Step 6 processes $2\hat{\subseteq}\wp 0$ with two effects. It is instantiated with 2 to give the proposition $2 \in 2 \to 2 \in \wp 0$. It also produces the following new shallow rule:

$$x | x \in 2 \Rightarrow x \in \wp 0 \tag{2.15}$$

This new shallow rule is triggered by $0 \in 2$ to produce the proposition $0 \in \wp 0$. Step 7 processes $0 \in \wp 0$ triggering Shallow Rule 1.5 to produce the proposition $\wp 0 \notin 0$ and triggering Shallow Rule 2.11 to produce the proposition $0 \hat{\subseteq} 0$. Step 8 processes $\wp 0 \notin 0$ making it available for later use (see Step 15) and Step 9 processes $0 \hat{\subseteq} 0$ making it available for later use.

Step 11 processes $2 \in 2 \to 2 \in \wp 0$ producing propositions $2 \notin 2$ and $2 \in \wp 0$. Step 12 processes $2 \in \wp 0$ triggering Shallow Rule 2.11 to produce $2 \hat{\subseteq} 0$. Step 14 processes $2 \hat{\subseteq} 0$ producing Shallow Rule 2.16.

$$x|x \in 2 \Rightarrow x \in 0 \tag{2.16}$$

This new shallow rule is triggered by $0 \in 2$ producing the proposition $0 \in 0$. Step 15 processes $0 \in 0$ causing three relevant actions. Mating $0 \in 0$ with $\wp 0 \notin 0$ produces the disequation $0 \neq 0$. Shallow Rule 1.4 is triggered by $0 \in 0$ yielding a clause giving that if $0 \in 0$, then $0 \neq 0$. Shallow Rule 1.5 is triggered by $0 \in 0$ to produce the proposition $0 \notin 0$. Step 16 processes $0 \neq 0$ yielding a clause indicating $0 = 0$ is true. Combining this with Step 15 we now know $0 \notin 0$ is true. Step 17 processes $0 \notin 0$ making it available for later use.

Step 24 processes $2 \notin 2$ (from Step 11) triggering Shallow Rule 1.14 to produce the proposition $\neg(2 = 0 \vee 2 = 1)$. Step 25 processes $\neg(2 = 0 \vee 2 = 1)$ producing the proposition $2 \neq 1$. Step 26 processes $2 \neq 1$ and a confrontation with $0 = 1$ produces the disequation $1 \neq 1$. Step 27 processes $0 \neq 0$ yielding a clause saying $0 = 0$ is true.

Step 28 processes the instantiation 1 producing propositions $1 \in 0 \to 1 \in 0$ and $1 \in 2 \to 1 \in \wp 0$ as well as a clause saying $1 \in 2 \to 1 \in \wp 0$ is true (since we know $2 \hat{\subseteq} \wp 0$ from Step 1). Step 29 processes $1 \in 0 \to 1 \in 0$ producing the proposition $1 \in 0$. Step 30 processes $1 \in 0$ triggering Shallow Rule 1.5 to produce the proposition $0 \notin 1$.

Step 32 processes $1 \in 2 \to 1 \in \wp 0$ producing the propositions $1 \notin 2$ and $1 \in \wp 0$ as well as a clause implying if $1 \in 2$ is true, then $1 \in \wp 0$ must be true. Step 33 processes $1 \in \wp 0$ triggering Shallow Rule 2.11 to produce the proposition $1 \hat{\subseteq} 0$ and a clause saying if $1 \in \wp 0$ is true, then $1 \hat{\subseteq} 0$ must be true. Step 34 processes $1 \hat{\subseteq} 0$ producing Shallow Rule 2.17.

$$x|x \notin 0 \Rightarrow x \notin 1 \tag{2.17}$$

This new shallow rule is triggered by $0 \notin 0$ producing a clause implying if $1 \hat{\subseteq} 0$, then $0 \notin 1$ (since we know $0 \notin 0$ as of Step 16).

At Step 64 Discriminator enters the closing phase.

Step 76 processes $0 \notin 1$ triggering Shallow Rule 1.9 to produce a clause that implies $0 \in 1$ is true.

Step 77 processes $1 \notin 2$ triggering Shallow Rule 1.14 to produce the proposition $\neg(1 = 0 \vee 1 = 1)$ and a clause indicating that either $1 \in 2$ is true or $1 = 0 \vee 1 = 1$ is false. The final step, Step 78, processes $\neg(1 = 0 \vee 1 = 1)$ yielding a clause that implies $1 = 0 \vee 1 = 1$ is true. The set of clauses is now unsatisfiable, as we now informally justify. Since $1 = 0 \vee 1 = 1$ is true, Step 77 gives $1 \in 2$ is true. By Step 32 $1 \in \wp 0$ must be true. By Step 33 $1 \hat{\subseteq} 0$ must be true. By Step 34 $0 \notin 1$ must be true, contradicting Step 76.

0:	2			
1:	14	Shallow Rule 2.13	18	-14 18 15
		Shallow Rule 1.11	19	
2:	19		20	
3:	20			
6:	18		23	
			Shallow Rule 2.15	
			24	
7:	24	Shallow Rule 1.5	-26	
		Shallow Rule 2.11	27	
8:	-26			
9:	27			
11:	23		-31 30	
12:	30	Shallow Rule 2.11	36	
14:	36		Shallow Rule 2.16	
			38	
15:	38	Mating	-21	
		Shallow Rule 1.4		-38 -21 -11
		Shallow Rule 1.5	-38	
16:	-21			21
17:	-38			
24:	-31	Shallow Rule 1.14	-43	
25:	-43		-44	
26:	-44	Confrontation	-46	
27:	-46			46
28:	1		52 53	-18 53
29:	52		54	
30:	54	Shallow Rule 1.5	-57	
32:	53		-59 58	-53 -59 58
33:	58	Shallow Rule 2.11	62	-58 62 -13
34:	62		Shallow Rule 2.17	38 -57 -62
76:	-57	Shallow Rule 1.9		57 -21 -8
77:	-59	Shallow Rule 1.14	-79	59 -79 -7
78:	-79			79 -46

Figure 2.3: Search Steps Leading to Proof of Subgoal 1(a) of Theorem 1

$\boxed{14}$	$0 \in 2$	$\boxed{16}$	$0 = 0 \vee 0 = 1$	$\boxed{17}$	$0 = 0$

Table 2.4: Propositions in Subgoal 1(b) of Theorem 1 Search

1:	$\boxed{-14}$	Shallow Rule 1.14	$\boxed{-16}$	$\boxed{14} \mid \boxed{-16} \mid \boxed{-7}$
2:	$\boxed{-16}$		$\boxed{-17}$	$\boxed{16} \mid \boxed{-17}$
3:	$\boxed{-17}$			$\boxed{17}$

Figure 2.4: Search Steps Leading to Proof of Subgoal 1(b) of Theorem 1

Subgoal 1b (Proving Cut Formula)

The next subgoal to prove corresponds to proving the cut formula $0 \in 2$. In principle DISCRIMINATOR is allowed to use the assumption

$$\neg \exists Y \in 2.2 \hat{\not\subseteq} \wp Y$$

to prove $0 \in 2$, although in practice it did not. The proof DISCRIMINATOR finds is the obvious, straightforward proof of $0 \in 2$ using Axiom 1.6. The only formulas used in the proof are in Table 2.4 and the relevant steps in the proof are in Figure 2.4. Before the search begins MiniSat is given unit clauses corresponding to the assumptions of the subgoal. In this case the only unit clauses leading to the proof are $\boxed{7}$ (for Axiom 1.6) and $\boxed{-14}$ (corresponding to the assumption $0 \notin 2$). The only unused step in the proof is Step 0 (not shown) processing Axiom 1.1.

In Step 1 DISCRIMINATOR processes $0 \notin 2$ and Shallow Rule 1.14 generates the proposition $\neg(0 = 0 \vee 0 = 1)$ and a clause implying $0 = 0 \vee 0 = 1$ is false. In Step 2 DISCRIMINATOR processes $\neg(0 = 0 \vee 0 = 1)$ generating the proposition $0 \neq 0$ and a clause implying $0 = 0$ is false. In Step 3 $0 \neq 0$ is processed giving a clause meaning $0 = 0$ is true, leading to unsatisfiability.

2.2.2 Subgoal 2

When considering the second subgoal the opening phase can break the problem down further after expanding atleast3, atleast2 and $\subseteq$. As a consequence the opening creates four fresh eigenvariables Y, a, b and c. In addition to the axioms (with abbreviations expanded), the search includes the following propositions:

1. $Y \hat{\subseteq} 2$
2. $a \in 2$,
3. $b \in Y$,

13	$a \in 2$	14	$b \in Y$
15	$c \in Y$	18	$c \in \wp b$
19	$a \in Y$	21	$c \hat{\subseteq} b$
22	$b = c$	26	$b \in 2$
27	$b = a$	28	$Y = Y$
33	$b = 0 \vee b = 1$	34	$c = a$
35	$c = c$	38	$c \in 2$
47	$a = c$	50	$a = 0 \vee a = 1$
52	$a = 0$	55	$b = 1$
56	$c = 1$	58	$b = 0$
59	$c = 0$	61	$c = 0 \vee c = 1$
63	$c = b$	64	$d \in c \to d \in b$
65	$d \in c$	66	$d \in b$
70	$d = d$		

Table 2.5: Propositions in Subgoal 2 of Theorem 1 Search

4. $c \in Y$,
5. $c \notin \wp b$ and
6. $a \notin Y$.

The proposition $Y \hat{\subseteq} 2$ produces the following shallow rule:

$$x | x \in Y \Rightarrow x \in 2 \tag{2.18}$$

The other propositions above do not produce a shallow rule and so they are given priority when the search begins. Other shallow rules that are produced from the axioms (with $\subseteq$ expanded) and are used in the search are Shallow Rule 1.11 and 2.2. The important propositions used in the search are given in Table 2.5 and the steps leading to a proof are shown in Figure 2.5. MiniSat is initially given unit clauses corresponding to the axioms and assumptions of the subgoal. These unit clauses are 17 positive unit clauses $\boxed{i}$ for $i \in \{1, \ldots, 17\}$ and the two negative unit clauses $\boxed{-18}$ and $\boxed{-19}$.

Step 1 processes $c \notin \wp b$ triggering Shallow Rule 2.12 to produce the formula $c \hat{\nsubseteq} b$ and a clause indicating $c \hat{\nsubseteq} b$ is true.

Step 3 processes $b \in Y$ mating it with $c \notin \wp b$ to produce the disequation $b \neq c$. The proposition $b \in Y$ also triggers Shallow Rule 2.18 producing the proposition $b \in 2$ and a clause indicating $b \in 2$ is true.

Step 4 processes $a \notin Y$ mating it with $b \in Y$ to produce disequations $b \neq a$ and $Y \neq Y$ and a clause meaning one of these disequations must be true. Step 5 processes $Y \neq Y$ giving a unit clause indicating $Y = Y$ is true. Hence we now know $b \neq a$ is true.

Step 6 processes $b \in 2$ triggering Shallow Rule 1.11 producing the proposition $b = 0 \vee b = 1$ and a clause meaning $b = 0 \vee b = 1$ is true.

Step 7 processes $c \in Y$ mating it with $a \notin Y$ to produce the disequation $c \neq a$ and a clause meaning $c \neq a$ is true. The proposition $c \in Y$ is also mated with $c \notin \wp b$

producing the disequation $c \neq c$. Finally $c \in Y$ triggers Shallow Rule 2.18 producing the proposition $c \in 2$ and a clause indicating $c \in 2$ is true. Step 8 processes $c \neq c$ yielding a unit clause indicating $c = c$ is true.

Step 11 processes $b \neq c$ making it available for future confrontations.

Step 13 processes $a \in 2$ mating it with $c \notin \wp b$ producing the disequation $a \neq c$ and triggering Shallow Rule 1.11 producing the proposition $a = 0 \vee a = 1$ and a clause meaning $a = 0 \vee a = 1$ is true. Step 15 processes $a = 0 \vee a = 1$ producing propositions $a = 0$ and $a = 1$ and a clause indicating one of the two is true. Step 18 processes $a = 0$ making it available for later confrontations.

Step 19 processes $c \in 2$ triggering Shallow Rule 1.11 producing the proposition $c = 0 \vee c = 1$ and a clause meaning $c = 0 \vee c = 1$ is true. Step 21 processes $c = 0 \vee c = 1$ producing propositions $c = 0$ and $c = 1$ and a clause indicating one of the two is true. Step 22 processes $c = 1$ making it available for later confrontations. Step 23 processes $c = 0$ making it available for later confrontations.

Step 24 processes $b = 0 \vee b = 1$ producing propositions $b = 0$ and $b = 1$ and a clause indicating one of the two is true. Step 25 processes $b = 0$ making it available for later confrontations.

Step 27 processes $c \hat{\not\subseteq} b$ (from Step 1). This creates a fresh eigenvariable d and produces the proposition $\neg(d \in c \to d \in b)$ and a clause indicating $\neg(d \in c \to d \in b)$ is true. Step 28 processes $\neg(d \in c \to d \in b)$ producing propositions $d \in c$ and $d \notin b$ and clauses indicating these propositions are true. Step 29 processes $d \notin b$ making it avaliable for matings. Step 30 processes $d \in c$ mating it with $d \notin b$ producing the disequation $d \neq d$ and a clause meaning either $c \neq b$ or $d \neq d$. Step 31 processes $d \neq d$ yielding the unit clause indicating $d = d$ is true. Hence we now know $c \neq b$ is true.

At Step 64 Discriminator enters the closing phase. By this point we know $b \neq a$ (Steps 4 and 5), $c \neq a$ (Step 7) and $c \neq b$ (Steps 30 and 31). We also know either $a = 0$ or $a = 1$ (Step 15), either $b = 0$ or $b = 1$ (Step 24) and either $c = 0$ or $c = 1$ (Step 21). These three disjunctions yield 8 cases, each of which is ruled out by the three disequations. The proof will be completed by a sequence of confrontations that communicate via propositional clauses that each of the 8 cases is ruled out.

Step 73 processes $b \neq a$ and is confronted by $a = 0$ yielding a clause saying that if $a = 0$, then $b \neq 0$. Step 89 processes $a \neq c$ (produced in Step 13, but different from $c \neq a$ produced in Step 7) and is confronted by $c = 1$ (yielding a clause saying if $c \neq a$ and $c = 1$, then $a \neq 1$) and $c = 0$ (yielding a clause saying if $c = 0$, then $a \neq 0$). Step 101 processes $c \neq b$ and is confronted by $c = 1$ (yielding a clause saying if $c = 1$, then $b \neq 1$) and $b = 1$ (yielding a clause saying if $b = 1$, then $c \neq 1$). Step 131 processes $b = 1$ and confronts $b \neq a$ (yielding a clause saying if $b = 1$, then $a \neq 1$) and $b \neq c$ (yielding a clause saying if $b = 1$, then $c \neq 1$). The final step, Step 152, processes $c \neq a$ and is confronted by $c = 0$ and $c = 1$. The result of the first confrontation is as expected and yields a clause saying if $c = 0$, then $a \neq 0$. The confrontation by $c = 1$ in principle adds a clause saying if $c = 1$, then $a \neq 1$, but this is not the clause that was reported as part of the minimally unsatisfiable set (reported by PicoMus [3] after MiniSat reported unsatisfiability). Instead the clause produced by the last confrontation says if $c = 1$, then $a \neq c$. (This is certainly true, since we know $c \neq a$ already and so $a \neq c$ is true independent of whether or not $c = 1$.) Combining this last clause with the clause from Step 89 we know if $c = 1$, then $a \neq 1$. At this point the clauses are sufficient to rule out every case.

1:	-18	Shallow Rule 2.12	-21	18 ‖ -21 ‖ -17
3:	14	Mating	-22	
		Shallow Rule 2.18	26	-14 ‖ 26 ‖ -12
4:	-19	Mating	-27 -28	-14 ‖ 19 ‖ -28 ‖ -27
5:	-28			28
6:	26	Shallow Rule 1.11	33	-26 ‖ 33 ‖ -7
7:	15	Mating	-34	-15 ‖ 19 ‖ -28 ‖ -34
			-35	
		Shallow Rule 2.18	38	-15 ‖ 38 ‖ -12
8:	-35			35
11:	-22			
13:	13	Mating	-47	
		Shallow Rule 1.11	50	-13 ‖ 50 ‖ -7
15:	50		52	-50 ‖ 52 ‖ 51
18:	52			
19:	38	Shallow Rule 1.11	61	-38 ‖ 61 ‖ -7
21:	61		59 56	-61 ‖ 59 ‖ 56
22:	56			
23:	59			
24:	33		58 55	-33 ‖ 58 ‖ 55
25:	58	Confrontation	-63	22 ‖ -58 ‖ -58 ‖ -59
27:	-21		-64	21 ‖ -64
28:	-64		65 -66	64 ‖ 65
				64 ‖ -66
29:	-66			
30:	65	Mating	-70	-65 ‖ 66 ‖ -63 ‖ -70
31:	-70			70
73:	-27	Confrontation		27 ‖ -52 ‖ -58 ‖ -52
89:	-47	Confrontation		47 ‖ -56 ‖ -51 ‖ -35
101:	-63	Confrontation		63 ‖ -59 ‖ -35 ‖ -22
				63 ‖ -56 ‖ -35 ‖ -22
131:	55	Confrontation		27 ‖ -55 ‖ -55 ‖ -51
				22 ‖ -55 ‖ -55 ‖ -56
152:	-34	Confrontation		34 ‖ -59 ‖ -59 ‖ -52
				34 ‖ -56 ‖ -35 ‖ -47

Figure 2.5: Search Steps Leading to Proof of Subgoal 2 of Theorem 1

Chapter 3

Theorem 3: $3 \setminus \{1\}$ has exactly 2 elements

The opening phase expands abbreviations including $\subseteq$. Due to this expansion the modified Axiom 1.1 yields only the shallow rule

$$x, y | x \neq y \Rightarrow x \hat{\not\subseteq} y, y \hat{\not\subseteq} x \tag{3.1}$$

and the modified Axiom 1.9 yields only the shallow rules

$$x, y | y \in \wp x \Rightarrow y \hat{\subseteq} x \tag{3.2}$$

$$x, y | y \notin \wp x \Rightarrow y \hat{\not\subseteq} x \tag{3.3}$$

The shallow rules produced by the other axioms are unchanged.

3.1 Subgoal 1

In the second subgoal we must prove $3 \setminus \{1\}$ has at least 2 elements. After expanding definitions the main search phase begins with the usual axioms (with $\subseteq$ expanded) and the following extra assumption:

- $\boxed{\text{-14}}$ $\neg \exists x. x \in 3 \setminus \{1\} \wedge 3 \setminus \{1\} \hat{\not\subseteq} \wp x$

The formulas used in the proof are given in Table 3.1 and the steps in the proof are given in Figures 3.1 and 3.2.

The extra assumption and all of the axioms produce shallow rules and so all are given low priority to be processed. It turns out the two shallow rules produced by the extra assumption are not used in the proof, so we will not make them explicit here. Since all the initial propositions are given low priority, Discriminator begins by processing the initial subterms of the subgoal as instantiations. There are seven such subterms (0, 1, 2, 3, 4, $\{1\}$ and $3 \setminus \{1\}$) and they are processed in the first seven steps. The instantiations used in the proof are 0, 1, 2, 3 and 4. (Note that 0, 2 and 4

$\boxed{11}$	$\forall x. x \notin x$	$\boxed{14}$	$\exists x. x \in 3 \setminus \{1\} \wedge 3 \setminus \{1\} \hat{\nsubseteq} \wp x$
$\boxed{19}$	$0 \in 3 \setminus \{1\} \wedge 3 \setminus \{1\} \hat{\nsubseteq} \wp 0$	$\boxed{20}$	$1 \in 3 \setminus \{1\} \wedge 3 \setminus \{1\} \hat{\nsubseteq} \wp 1$
$\boxed{22}$	$3 \setminus \{1\} \hat{\subseteq} \wp 0$	$\boxed{23}$	$0 \in 3 \setminus \{1\}$
$\boxed{26}$	$3 \setminus \{1\} \hat{\subseteq} \wp 1$	$\boxed{28}$	$2 \in 3 \setminus \{1\} \to 2 \in \wp 0$
$\boxed{29}$	$4 \in 3 \setminus \{1\} \to 4 \in \wp 0$	$\boxed{32}$	$0 \in 3 \setminus \{1\} \to 0 \in \wp 0$
$\boxed{33}$	$1 \in 3 \setminus \{1\} \to 1 \in \wp 0$	$\boxed{34}$	$3 \in 3 \setminus \{1\} \to 3 \in \wp 0$
$\boxed{35}$	$3 \in \wp 0$	$\boxed{37}$	$4 \in \wp 0$
$\boxed{39}$	$2 \in \wp 0$	$\boxed{40}$	$2 \in 3 \setminus \{1\}$
$\boxed{43}$	$2 \hat{\subseteq} 0$	$\boxed{46}$	$4 \hat{\subseteq} 0$
$\boxed{47}$	$2 \in 4 \to 2 \in 0$	$\boxed{51}$	$0 \in 4 \to 0 \in 0$
$\boxed{52}$	$1 \in 4 \to 1 \in 0$	$\boxed{53}$	$3 \in 4 \to 3 \in 0$
$\boxed{54}$	$1 \in 0$	$\boxed{56}$	$0 = 1$
$\boxed{57}$	$0 \in 1$	$\boxed{65}$	$3 \in 0$
$\boxed{68}$	$0 \in 3$	$\boxed{69}$	$0 \in 0$
$\boxed{71}$	$0 = 0$	$\boxed{82}$	$2 \in 0$
$\boxed{85}$	$0 \in 2$	$\boxed{86}$	$2 \in 2 \to 2 \in 0$
$\boxed{98}$	$2 \in 2$	$\boxed{101}$	$3 \hat{\subseteq} 0$
$\boxed{102}$	$2 \in 3 \to 2 \in 0$	$\boxed{111}$	$2 \in 3$
$\boxed{115}$	$1 \in \wp 0$	$\boxed{118}$	$1 \hat{\subseteq} 0$
$\boxed{124}$	$1 \in 1 \to 1 \in 0$	$\boxed{130}$	$1 \in 1$
$\boxed{132}$	$0 \in \wp 0$	$\boxed{134}$	$0 \hat{\subseteq} 0$
$\boxed{135}$	$2 \in 0 \to 2 \in 0$	$\boxed{140}$	$1 \in 0 \to 1 \in 0$
$\boxed{147}$	$1 = 1$	$\boxed{149}$	$2 = 1$
$\boxed{181}$	$2 \in \{1\}$		
$\boxed{199}$	$0 \in 3 \setminus \{1\} \to 0 \in \wp 1$	$\boxed{207}$	$2 = 2$
$\boxed{208}$	$0 \in \wp 1$	$\boxed{214}$	$0 \hat{\subseteq} 1$
$\boxed{269}$	$0 \in 3 \wedge 0 \notin \{1\}$	$\boxed{331}$	$0 = 0 \vee 0 = 1 \vee 0 = 2$
$\boxed{332}$	$0 = 0 \vee 0 = 1$	$\boxed{365}$	$2 \in 3 \wedge 2 \notin \{1\}$
$\boxed{411}$	$2 = 0 \vee 2 = 1$	$\boxed{412}$	$2 = 0 \vee 2 = 1 \vee 2 = 2$

Table 3.1: Propositions in Subgoal 1 of Theorem 3 Search

are used in the proof even though they did not appear in the conjecture about $3 \setminus \{1\}$. They only appear in the axioms.)

Step 7 must process one of the low priority propositions. It processes the extra assumption

$$\neg \exists x.x \in 3 \setminus \{1\} \wedge 3 \setminus \{1\} \hat{\not\subseteq} \wp x$$

instantiating it with the terms processed above. Two of these instantiations will play a role in the proof: the instance with 0 (which is reasonable since $0 \in 3 \setminus \{1\}$ and $3 \setminus \{1\} \hat{\not\subseteq} \wp 0$ are actually true) and the instance with 1 (which is not very meaningful, but ultimately produces propositions that will be useful). A clause is produced saying the instance with 0 is false. Step 8 processes the instance with 0 giving the propositions $0 \notin 3 \setminus \{1\}$ and $3 \setminus \{1\} \hat{\subseteq} \wp 0$ and a clause indicating one of these two propositions must be true. Step 10 processes the instance with 1 producing the proposition $3 \setminus \{1\} \hat{\subseteq} \wp 1$, a proposition that will be processed in Step 99.

Step 11 processes $3 \setminus \{1\} \hat{\subseteq} \wp 0$ instantiating it with 0, 1, 2, 3 and 4 producing the propositions $0 \in 3 \setminus \{1\} \to 0 \in \wp 0$, $1 \in 3 \setminus \{1\} \to 1 \in \wp 0$, $2 \in 3 \setminus \{1\} \to 2 \in \wp 0$, $3 \in 3 \setminus \{1\} \to 3 \in \wp 0$ and $4 \in 3 \setminus \{1\} \to 4 \in \wp 0$. A clause is produced that says $2 \in 3 \setminus \{1\} \to 2 \in \wp 0$ is true if $3 \setminus \{1\} \hat{\subseteq} \wp 0$ is true. The instance with 2 is the most important since $2 \in 3 \setminus \{1\}$ and $2 \notin \wp 0$ are true, making the corresponding implication false.

Step 12 processes $3 \in 3 \setminus \{1\} \to 3 \in \wp 0$ producing the proposition $3 \in \wp 0$. Step 13 processes $4 \in 3 \setminus \{1\} \to 4 \in \wp 0$. producing the proposition $4 \in \wp 0$.

Step 14 processes $2 \in 3 \setminus \{1\} \to 2 \in \wp 0$ producing the propositions $2 \notin 3 \setminus \{1\}$ and $2 \in \wp 0$ and a clause indicating that if $2 \in 3 \setminus \{1\} \to 2 \in \wp 0$ is true, then either $2 \notin 3 \setminus \{1\}$ or $2 \in \wp 0$ must be true. Step 15 processes $2 \in \wp 0$ triggering Shallow Rule 3.2 to produce the proposition $2 \hat{\subseteq} 0$ and a clause meaning that $2 \in \wp 0$ implies $2 \hat{\subseteq} 0$.

Step 16 processes $4 \in \wp 0$ triggering Shallow Rule 3.2 to produce the proposition $4 \hat{\subseteq} 0$. Step 17 processes $4 \hat{\subseteq} 0$ instantiating it with 0, 1, 2 and 3 producing propositions $0 \in 4 \to 0 \in 0$, $1 \in 4 \to 1 \in 0$, $2 \in 4 \to 2 \in 0$ and $3 \in 4 \to 3 \in 0$. Step 18 processes $1 \in 4 \to 1 \in 0$ producing the proposition $1 \in 0$. Step 19 processes $1 \in 0$ triggering Shallow Rule 1.4 to produce the disequation $0 \neq 1$ and Shallow Rule 1.5 to produce the proposition $0 \notin 1$. Step 24 processes $3 \in 4 \to 3 \in 0$ producing the proposition $3 \in 0$. Step 25 processes $3 \in 0$ triggering Shallow Rule 1.5 to produce the proposition $0 \notin 3$. Step 26 processes $0 \in 4 \to 0 \in 0$ producing the proposition $0 \in 0$. Step 27 processes $0 \in 0$ triggering Shallow Rule 1.4 to produce the disequation $0 \neq 0$ and yielding a clause saying $0 = 0$ implies $0 \notin 0$. The propostion $0 \in 0$ also triggers Shallow Rule 1.5 to produce the proposition $0 \notin 0$. Step 28 processes $0 \neq 0$ yielding a clause recording that $0 = 0$ is true. Note that this also implies $0 \notin 0$ when combined with the clause produced in Step 27. Step 29 processes $0 \notin 0$ making it available for later use (see Step 34). Step 32 processes $2 \in 4 \to 2 \in 0$ producing the proposition $2 \in 0$. Step 33 processes $2 \in 0$ triggering Shallow Rule 1.5 to produce the proposition $0 \notin 2$.

Step 34 processes $2 \hat{\subseteq} 0$ with two effects. The proposition is instantiated with 2 producing the proposition $2 \in 2 \to 2 \in 0$. In addition the following new shallow rule is

0:	3				
1:	1				
2:	0				
5:	4				
6:	2				
7:	-14		-19		14 ‖ -19
			-20		
8:	-19		-23	22	19 ‖ -23 ‖ 22
10:	-20		26		
11:	22		28		-22 ‖ 28
			29	32	
			33	34	
12:	34		35		
13:	29		37		
14:	28		-40	39	-28 ‖ -40 ‖ 39
15:	39	Shallow Rule 3.2	43		-39 ‖ 43 ‖ -13
16:	37	Shallow Rule 3.2	46		
17:	46		47	51	
			52	53	
18:	52		54		
19:	54	Shallow Rule 1.4	-56		
		Shallow Rule 1.5	-57		
24:	53		65		
25:	65	Shallow Rule 1.5	-68		
26:	51		69		
27:	69	Shallow Rule 1.4	-71		-69 ‖ -71 ‖ -11
		Shallow Rule 1.5	-69		
28:	-71				71
29:	-69				
32:	47		82		
33:	82	Shallow Rule 1.5	-85		
34:	43		86		
			Shallow Rule 3.4		69 ‖ -85 ‖ -43
41:	86		-98		
42:	35	Shallow Rule 3.2	101		
43:	101		102		
46:	102		-111		

Figure 3.1: Search Steps Leading to Proof of Subgoal 1 of Theorem 3 (Part 1)

created:

$$x|x \notin 0 \Rightarrow x \notin 2 \tag{3.4}$$

This new shallow rule is triggered by $0 \notin 0$ (processed in Step 29) yielding a clause that says $2\hat{\subseteq}0$ and $0 \in 2$ imply $0 \in 0$. Step 41 processes $2 \in 2 \rightarrow 2 \in 0$ producing the proposition $2 \notin 2$.

Step 42 processes $3 \in \wp 0$ (produced in Step 12) triggering Shallow Rule 3.2 to produce $3\hat{\subseteq}0$. Step 43 processes $3\hat{\subseteq}0$ instantiating it with 2 to produce $2 \in 3 \rightarrow 2 \in 0$. Step 46 processes $2 \in 3 \rightarrow 2 \in 0$ producing $2 \notin 3$. We will return to process $2 \notin 3$ in Step 234.

Step 51 processes $1 \in 3 \setminus \{1\} \rightarrow 1 \in \wp 0$ (one of the instances produced in Step 11) producing $1 \in \wp 0$. Step 52 processes $1 \in \wp 0$ triggering Shallow Rule 3.2 to produce $1\hat{\subseteq}0$. Step 53 processes $1\hat{\subseteq}0$ instantiating with 1 to produce $1 \in 1 \rightarrow 1 \in 0$. Step 59 processes $1 \in 1 \rightarrow 1 \in 0$ producing the formula $1 \notin 1$.

Step 61 processes $0 \in 3 \setminus \{1\} \rightarrow 0 \in \wp 0$ (that last of unprocessed instances produced in Step 11) producing $0 \in \wp 0$. Step 62 processes $0 \in \wp 0$ triggering Shallow Rule 3.2 to produce $0\hat{\subseteq}0$. Step 63 processes $0\hat{\subseteq}0$ instantiating with 1 and 2 to produce $1 \in 0 \rightarrow 1 \in 0$ and $2 \in 0 \rightarrow 2 \in 0$. Step 64 processes $1 \in 0 \rightarrow 1 \in 0$ producing $1 \notin 0$. Step 68 processes $1 \notin 0$ mating it with $1 \in \wp 0$ (processed in Step 15) and $2 \in 0$ (processed in Step 33) producing disequations $1 \neq 1$ and $2 \neq 1$. Step 69 processes $1 \neq 1$ yielding a unit clause indicating $1 = 1$ is true.

Step 76 processes $2 \in 0 \rightarrow 2 \in 0$ producing $2 \notin 0$.

Step 99 processes $3 \setminus \{1\}\hat{\subseteq}\wp 1$ (produced in Step 10) instantiating it with 0 to produce $0 \in 3 \setminus \{1\} \rightarrow 0 \in \wp 1$.

Step 100 processes $2 \notin 0$ mating it with $2 \in \wp 0$ to produce the disequation $2 \neq 2$. Step 101 processes $2 \neq 2$ yielding a unit clause indicating $2 = 2$ is true.

Step 102 processes $0 \in 3 \setminus \{1\} \rightarrow 0 \in \wp 1$ producing $0 \in \wp 1$. Step 106 processes $0 \in \wp 1$ triggering Shallow Rule 3.2 to produce $0\hat{\subseteq}1$. Step 107 processes $0\hat{\subseteq}1$ producing the following new shallow rule:

$$x|x \in 0 \Rightarrow x \in 1 \tag{3.5}$$

This new shallow rule is triggered by $0 \in 0$ (processed in Step 27) and $1 \in 0$ (processed in Step 19) to produce propositions $0 \in 1$ and $1 \in 1$. Step 113 processes $1 \in 1$ triggering Shallow Rule 1.5 to produce a clause essentially saying $1 \notin 1$ is true.

At Step 128 DISCRIMINATOR enters the closing phase. During this phase only propositions with an outermost propositional connective will be added to the priority queue.

Step 150 processes $0 \notin 3 \setminus \{1\}$ (produced in Step 8) triggering Shallow Rule 1.37 to produce $\neg(0 \in 3 \wedge 0 \notin \{1\})$ and a clause that says if $0 \in 3 \wedge 0 \notin \{1\}$, then $0 \in 3 \setminus \{1\}$.

Step 163 processes $1 \notin 1$ making it avialable for use later.

Step 180 processes $\neg(0 \in 3 \wedge 0 \notin \{1\})$ yielding a clause that says $0 \in 3$ and $0 \notin \{1\}$ imply $0 \in 3 \wedge 0 \notin \{1\}$

51:	33		115	
52:	115	Shallow Rule 6bb6	118	
53:	118		124	
59:	124		-130	
61:	32		132	
62:	132	Shallow Rule 6bb6	134	
63:	134		135 140	
64:	140		-54	
68:	-54	Mating	-147 -149	
69:	-147			147
76:	135		-82	
99:	26		199	
100:	-82	Mating	-207	
101:	-207			207
102:	199		208	
106:	208	Shallow Rule 6bb6	214	
107:	214		Shallow Rule 29ef	
			57 130	
113:	130	Shallow Rule 9e8c		-130 ‖ -130 ‖ -10
150:	-23	Shallow Rule a9eb	-269	23 ‖ -269 ‖ -1
163:	-130			
180:	-269			269 ‖ -68 ‖ 64
189:	-56	Shallow Rule a7b0		56 ‖ -64 ‖ -4
195:	-40	Shallow Rule a9eb	-365	40 ‖ -365 ‖ -1
196:	-365			365 ‖ -111 ‖ 181
212:	-68	Shallow Rule 1f6a	-331	68 ‖ -331 ‖ -6
213:	-331		-332	331 ‖ -332
214:	-332			332 ‖ -71
219:	-57	Shallow Rule c3ed		57 ‖ -71 ‖ -8
227:	-85	Shallow Rule 43a9		85 ‖ -332 ‖ -7
230:	-149	Shallow Rule a7b0		149 ‖ -181 ‖ -4
232:	-98	Shallow Rule 43a9	-411	98 ‖ -411 ‖ -7
233:	-411			411 ‖ -149
234:	-111	Shallow Rule 1f6a	-412	111 ‖ -412 ‖ -6
235:	-412			412 ‖ -207
236:	57	Mating		-57 ‖ 130 ‖ -147 ‖ -56
300:	11			-11 ‖ -98

Figure 3.2: Search Steps Leading to Proof of Subgoal 1 of Theorem 3 (Part 2)

Step 189 processes $0 \neq 1$ triggering Shallow Rule refsr:singd to produce a clause saying $0 \in \{1\}$ implies $0 = 1$.

Step 195 processes $2 \notin 3 \setminus \{1\}$ triggering Shallow Rule 1.37 to produce $\neg(2 \in 3 \wedge 2 \notin \{1\})$ and a clause saying $2 \in 3 \wedge 2 \notin \{1\}$ implies $2 \in 3 \setminus \{1\}$. Step 196 processes $\neg(2 \in 3 \wedge 2 \notin \{1\})$ yielding a clause saying $2 \in 3$ and $2 \notin \{1\}$ imply $2 \in 3 \wedge 2 \notin \{1\}$.

Step 212 processes $0 \notin 3$ triggering Shallow Rule 1.19 producing $\neg(0 = 0 \vee 0 = 1 \vee 0 = 2)$ and a clause saying $0 = 0 \vee 0 = 1 \vee 0 = 2$ implies $0 \in 3$. Step 213 processes $\neg(0 = 0 \vee 0 = 1 \vee 0 = 2)$ producing $\neg(0 = 0 \vee 0 = 1)$ and a clause saying $0 = 0 \vee 0 = 1$ implies $0 = 0 \vee 0 = 1 \vee 0 = 2$. Step 214 processes $\neg(0 = 0 \vee 0 = 1)$ producing a clause saying $0 = 0$ implies $0 = 0 \vee 0 = 1$. Recall we know $0 = 0$ from Step 28. Combining this with the information from Steps 212, 213 and 214 we now know $0 \in 3$ must be true.

Step 219 processes $0 \notin 1$ triggering Shallow Rule 1.9 to produce a clause saying $0 = 0$ implies $0 \in 1$. Since we know $0 = 0$ already we now know $0 \in 1$.

Step 227 processes $0 \notin 2$ triggering Shallow Rule 1.14 to produce a clause saying $0 = 0 \vee 0 = 1$ implies $0 \in 2$. We know $0 = 0 \vee 0 = 1$ from Step 214 and so we now know $0 \in 2$. Combining this with the information from Steps 34 and 28 we now know $2 \hat{\subseteq} 0$ must be false. By Step 15 we must have $2 \notin \wp 0$.

Step 230 processes $2 \neq 1$ triggering Shallow Rule 1.31 to produce a clause saying $2 \neq 1$ implies $2 \notin \{1\}$.

Step 232 processes $2 \notin 2$ triggering Shallow Rule 1.14 to produce $\neg(2 = 0 \vee 2 = 1)$ and a clause saying $2 = 0 \vee 2 = 1$ implies $2 \in 2$. Step 233 processes $\neg(2 = 0 \vee 2 = 1)$ producing a clause saying $\neg(2 = 0 \vee 2 = 1)$ implies $2 \neq 1$.

Step 234 processes $2 \notin 3$ triggering Shallow Rule 1.19 to produce the proposition $\neg(2 = 0 \vee 2 = 1 \vee 2 = 2)$ and a clause indicating $2 = 0 \vee 2 = 1 \vee 2 = 2$ implies $2 \in 3$. Step 235 processes $\neg(2 = 0 \vee 2 = 1 \vee 2 = 2)$ producing a clause saying $2 = 2$ implies $2 = 0 \vee 2 = 1 \vee 2 = 2$. We know $2 = 2$ from Step 101 and so we now know $2 \in 3$.

Step 236 processes $0 \in 1$ mating it with $1 \in 1$ to produce a clause saying that if $0 \in 1$ and $1 \notin 1$, then either $1 \neq 1$ or $0 \neq 1$. We know $0 \in 1$ from Step 219, $1 \notin 1$ from Step 113 and $1 = 1$ from Step 69. Hence the clause implies $0 \neq 1$. Combining this with Step 189 we know $0 \notin \{1\}$. Combining this with Step 180 and $0 \in 3$, we know $0 \in 3 \wedge 0 \notin \{1\}$. Combining this with Step 150 we now know $0 \in 3 \setminus \{1\}$. Combining this with Step 8 we must have $3 \setminus \{1\} \hat{\subseteq} \wp 0$. Working forward from this, Step 11 implies $2 \in 3 \setminus \{1\} \to 2 \in \wp 0$ must be true and Step 14 means either $2 \notin 3 \setminus \{1\}$ or $2 \in \wp 0$ must be true. We know $2 \notin \wp 0$ from Step 227. Hence we must have $2 \notin 3 \setminus \{1\}$.

The final step, Step 300, processes the proposition $\forall x.x \notin x$ (Axiom 1.2) instantiating it with 2 yielding a clause giving the truth of $2 \notin 2$. Combining this with Steps 232 and 233 we now know $\neg(2 = 0 \vee 2 = 1)$ and $2 \neq 1$. Combining this with Step 230 we know $2 \notin \{1\}$. Step 196 with $2 \in 3$ and $2 \notin \{1\}$ now give $2 \in 3 \wedge 2 \notin \{1\}$. This with Step 195 yields $2 \in 3 \setminus \{1\}$, contradicting what we concluded in Step 236.

3.2 Subgoal 2

In the second subgoal we must prove $3 \setminus \{1\}$ does not have at least 3 elements. After expanding definitions and traversing quantifiers creating eigenvariables Y, a, b and c (during the opening) the main search phase begins with the usual axioms (with $\subseteq$ expanded) assigned labels $\boxed{i}$ with $i \in \{1, \ldots, 11\} \cup \{16, 17\}$ and the following extra assumptions:

- $\boxed{12}$ $Y \hat{\subseteq} 3 \setminus \{1\}$
- $\boxed{13}$ $a \in 3 \setminus \{1\}$
- $\boxed{14}$ $b \in Y$
- $\boxed{15}$ $c \in Y$
- $\boxed{\text{-18}}$ $c \notin \wp b$
- $\boxed{\text{-19}}$ $a \notin Y$

The proof proceeds by confirming a, b and c are distinct members of $3 \setminus \{1\}$, leading to a contradiction. The assumption $\boxed{12}$ is only used in the form of the following shallow rule it produces:

$$x | x \in Y \Rightarrow x \in 3 \setminus \{1\} \tag{3.6}$$

The formulas used in the proof are given in Table 3.2 and the steps in the proof are given in Figures 3.3 and 3.4.

Step 0 processes $c \notin \wp b$ triggering Shallow Rule 3.3 to produce $c \hat{\nsubseteq} b$ and a clause indicating $c \hat{\nsubseteq} b$ is true. This new proposition will not be processed until Step 57.

Step 1 processes $a \notin Y$ making it available for later use (see Steps 14 and 61).

Step 4 processes proceses $a \in 3 \setminus \{1\}$ having multiple effects. First it mates with $a \notin Y$ producing disequations $a \neq a$ and $3 \setminus \{1\} \neq Y$. It additionally mates with $c \notin \wp b$ producing the disequation $a \neq c$. The proposition $a \in 3 \setminus \{1\}$ also triggers Shallow Rule 1.5 producing $3 \setminus \{1\} \notin a$ and triggers Shallow Rule 1.36 producing $a \in 3 \wedge a \notin \{1\}$ along with a clause saying this conjunction is true.

Step 5 processes $a \neq a$ yielding a unit clause saying $a = a$ is true.

Step 6 processes $3 \setminus \{1\} \notin a$ making it available for later use (see Step 69).

Step 7 processes $3 \setminus \{1\} \neq Y$ triggering Shallow Rule 1.31 producing the proposition $3 \setminus \{1\} \notin \{Y\}$. Step 8 processes $3 \setminus \{1\} \notin \{Y\}$ making it available for later use.

Step 14 processes $b \in Y$ mating it two relevant times. Mating with $3 \setminus \{1\} \notin a$ produces the disequation $Y \neq a$. Mating with $a \notin Y$ produces disequations $b \notin= a$ and $Y \neq Y$ and a clause indicating that one of these disequations must be true. The proposition $b \in Y$ also triggers Shallow Rule 3.6 producing proposition $b \in 3 \setminus \{1\}$ and a clause indicating $b \in 3 \setminus \{1\}$ is true. Step 15 processes $Y \neq Y$ yielding a unit clause recording the truth of $Y = Y$. Combining this with Step 14 we now know $b \neq a$ is true. Steps 16 and 19 process $b \neq a$ and $Y \neq a$ making them available for later use.

13	$a \in 3 \setminus \{1\}$	14	$b \in Y$
15	$c \in Y$	18	$c \in \wp b$
19	$a \in Y$	20	$c \hat{\subseteq} b$
21	$a = a$	22	$3 \setminus \{1\} = Y$
23	$a = c$	26	$3 \setminus \{1\} \in a$
27	$a \in 3 \wedge a \notin \{1\}$	30	$3 \setminus \{1\} \in \{Y\}$
43	$Y = a$	44	$b = a$
45	$Y = Y$	50	$b \in 3 \setminus \{1\}$
67	$b \in 3 \wedge b \notin \{1\}$	68	$b \in 3$
69	$b \in \{1\}$		
75	$3 = b$	76	$3 \in b$
77	$b = 0 \vee b = 1 \vee b = 2$		
83	$b = 2$	84	$b = 0 \vee b = 1$
88	$a = 2$	110	$b = 1$
111	$b = 0$	113	$a = 1$
126	$a = 0$	130	$b \in \{0\}$
139	$\{0\} = b$	149	$a \in 3$
150	$a \in \{1\}$	152	$a = 0 \vee a = 1 \vee a = 2$
153	$a = 0 \vee a = 1$	176	$d \in c \rightarrow d \in b$
177	$d \in c$	178	$d \in b$
180	$c = b$	185	$c = a$
188	$d = c$	192	$c = 1$
195	$c = 0$	198	$c = 2$
204	$c = c$	207	$c \in 3 \setminus \{1\}$
214	$c \in 3 \wedge c \notin \{1\}$	219	$c \in 3$
220	$c \in \{1\}$	223	$c = 0 \vee c = 1 \vee c = 2$
249	$c = 0 \vee c = 1$	270	$\{0\} \hat{\subseteq} b$
285	$e \in \{0\} \rightarrow e \in b$	286	$e \in \{0\}$
292	$e = 3 \setminus \{1\}$	301	$e \in \{3 \setminus \{1\}\}$
304	$d = e$	311	$d \in \{e\}$
321	$d = d$		

Table 3.2: Propositions in Subgoal 2 of Theorem 3 Search

Step 22 processes $b \in 3 \setminus \{1\}$ triggering Shallow Rule 1.36 to produce $b \in 3 \wedge b \notin \{1\}$ and a clause indicating the conjunction is true. Step 23 processes the conjunction producing $b \in 3$ and clauses indicating both $b \in 3$ and $b \notin \{1\}$ are true. Step 24 processes $b \in 3$ triggering three relevant shallow rules. Shallow Rule 1.4 produces $3 \neq b$, Shallow Rule 1.5 produces $3 \notin b$ and Shallow Rule 1.17 produces $b = 0 \vee b = 1 \vee b = 2$. Steps 25 and 26 process $3 \notin b$ and $3 \neq b$ making them available for later use (e.g., see Step 69). Step 27 processes $b = 0 \vee b = 1 \vee b = 2$ producing $b = 0 \vee b = 1$ Step 33 processes $b = 0 \vee b = 1$ producing $b = 0$ and $b = 1$. Note that at this point these equations are only produced as propositions to process, not as equations we have any reason to believe are true.

Step 34 processes $b = 1$ confronting $3 \neq b$ and $Y \neq a$ to produce disequations $b \neq 1$ and $a \neq 1$. In addition $b = 1$ triggers Shallow Rule 1.28 producing a clause indicating $b \neq 1$ is true (since we know $b \notin \{1\}$ from Step 23). (The proposition $b \in \{1\}$ will play a role in the ultimate propositional unsatisfiability, but does not need to be processed during the proof.)

Step 36 processes $b = 0$ triggering Shallow Rule 1.28 producing the proposition $b \in \{0\}$ (and a clause relating $b = 0$ and $b \in \{0\}$ that will play no role in the ultimate proof). Step 38 processes $b \in \{0\}$ mating with $3 \notin b$ to produce the disequation $\{0\} \neq b$.

Step 43 processes $a \in 3 \wedge a \notin \{1\}$ producing propositions $a \in 3$ and $a \notin \{1\}$ and clauses implying the two propositions are true. Step 44 processes $a \in 3$ triggering Shallow Rule 1.17 to produce the proposition $a = 0 \vee a = 1 \vee a = 2$. Step 45 processes $a = 0 \vee a = 1 \vee a = 2$ producing $a = 0 \vee a = 1$ and $a = 2$. Step 46 processes $a = 0 \vee a = 1$ producing $a = 0$ and $a = 1$. Step 47 processes $a = 0$ confronting $b \neq a$ yielding a clause saying if $a = 0$, then $b \neq 0$ (since we know $b \neq a$ from Step 15 and $a = a$ from Step 5). Step 56 processes $a = 1$ making it available for later use.

Step 57 processes $c \hat{\not\subseteq} b$ (produced in Step 0). This creates a fresh eigenvariable d and produces the proposition $\neg(d \in c \rightarrow d \in b)$ along with a clause meaning $d \in c \rightarrow d \in b$ is false. Step 58 processes $\neg(d \in c \rightarrow d \in b)$ producing $d \in c$ and $d \notin b$ along with clauses meaning these two propositions are true. Step 59 processes $d \in c$ mating it with $3 \notin b$, $3 \setminus \{1\} \notin a$ and $c \notin \wp b$ to produce disequations $c \neq b$, $c \neq a$ and $d \neq c$. Step 60 processes $d \neq c$ and is confronted by $a = 1$ producing disequation $c \neq 1$.

Step 61 processes $c \in Y$ with three effect. Mating with $a \in Y$ yields a clause from which we can infer $c \neq a$. Mating with $c \notin \wp b$ produces the disequation $c \neq c$. Finally the Shallow Rule 3.6 producing proposition $c \in 3 \setminus \{1\}$ and a clause indicating $c \in 3 \setminus \{1\}$ is true. Step 62 processes $c \neq c$ yielding a unit clause indicating $c = c$ is true. Step 65 processes $c \in 3 \setminus \{1\}$ triggering Shallow Rule 1.36 to produce $c \in 3 \wedge c \notin \{1\}$ and a clause indicating the conjunction is true. Step 68 processes the conjunction producing $c \in 3$ and $c \notin \{1\}$ and clauses indicating both $c \in 3$ and $c \notin \{1\}$ are true. Step 69 processes $c \in 3$ triggering Shallow Rule 1.17 to produce $c = 0 \vee c = 1 \vee c = 2$ and a clause indicating this disjunction is true. Step 70 processes $c \notin \{1\}$ triggering Shallow Rule 1.29 to produce a clause indicating $c \neq 1$ is true. Step 74 processes $c = 0 \vee c = 1 \vee c = 2$ producing $c = 0 \vee c = 1$ and $c = 2$. Step 75 processes $c = 0 \vee c = 1$ producing $c = 0$. Steps 76 and 82 process $c = 0$ and $c = 2$ making them available for

0:	-18	Shallow Rule 3.3	-20	18 ‖ -20 ‖ -17
1:	-19			
4:	13	Mating	-21 -22	
			-23	
		Shallow Rule 1.5	-26	
		Shallow Rule 1.36	27	-13 ‖ 27 ‖ -1
5:	-21			21
6:	-26			
7:	-22	Shallow Rule 1.31	-30	
8:	-30			
14:	14	Mating	-43	
			-44 -45	-14 ‖ 19 ‖ -45 ‖ -44
		Shallow Rule 3.6	50	-14 ‖ 50 ‖ -12
15:	-45			45
16:	-44			
19:	-43			
22:	50	Shallow Rule 1.36	67	-50 ‖ 67 ‖ -1
23:	67		68	-67 ‖ 68
				-67 ‖ -69
24:	68	Shallow Rule 1.4	-75	
		Shallow Rule 1.5	-76	
		Shallow Rule 1.17	77	
25:	-76			
26:	-75			
27:	77		84	
33:	84		111 110	
34:	110	Confrontation	-110 -113	
		Shallow Rule 1.28		-110 ‖ 69 ‖ -4
36:	111	Shallow Rule 1.28	130	
38:	130	Mating	-139	
43:	27		149 -150	-27 ‖ 149
				-27 ‖ -150
44:	149	Shallow Rule 1.17	152	
45:	152		153 88	
46:	153		126 113	
47:	126	Confrontation		44 ‖ -126 ‖ -111 ‖ -21
56:	113			

Figure 3.3: Search Steps Leading to Proof of Subgoal 2 of Theorem 3 (Part 1)

later use.

Step 84 processes $\{0\} \neq b$ triggering Shallow Rule 3.1 to produce proposition $\{0\}\hat{\not\subseteq}b$. Step 89 processes $\{0\}\hat{\not\subseteq}b$ creating an eigenvariable e and producing $\neg(e \in \{0\} \to e \in b)$. Step 90 processes $\neg(e \in \{0\} \to e \in b)$ producing $e \in \{0\}$. Step 91 processes $e \in \{0\}$ mating it with $3 \setminus \{1\} \in a$ to produce the disequation $e \neq 3 \setminus \{1\}$. Step 92 processes $e \neq 3 \setminus \{1\}$ triggering Shallow Rule 1.31 to produce $e \notin \{3 \setminus \{1\}\}$. Step 93 processes $e \notin \{3 \setminus \{1\}\}$ mating with $d \in c$ to produce the disequation $d \neq e$. Step 96 processes $d \neq e$ triggering Shallow Rule 1.31 to produce $d \notin \{e\}$. Step 97 processes $d \notin \{e\}$ mating with to produce the disequation $d \neq d$. Step 98 processes $d \neq d$ to yield a unit clause indicating $d = d$ is true.[1]

At Step 128 Discriminator enters the closing phase. During this phase only propositions with an outermost propositional connective will be added to the priority queue.

Step 169 processes $a \notin \{1\}$ (from Step 43) triggering Shallow Rule 1.29 to produce a clause indicating $a \neq 1$ is true. Step 172 processes $a \neq c$ and is confronted by $a = 0$ producing a clause implying if $a \neq c$ and $a = 0$ are true, then $c \neq 0$ must be true. Step 195 processes $a \neq 1$ triggering Shallow Rule 1.18 to produce a clause saying either $a = 0$ or $a = 2$ must be true. Step 202 processes $c \neq 1$ triggering Shallow Rule 1.18 to produce a clause saying either $c = 0$ or $c = 2$ must be true. Step 223 processes $c \neq b$ and is confronted by $c = 0$ and $c = 2$ giving clauses that say $c \neq b$ and $c = 0$ imply $b \neq 0$ and $c \neq b$ and $c = 2$ imply $b \neq 2$. (While $c \neq b$ easily follows from $d \in c$ and $d \notin b$, but Discriminator will not process $d \notin b$ until the final step.)

Step 244 processes $b \neq 1$ triggering Shallow Rule 1.18 to produce a clause saying either $b = 0$ or $b = 2$ must be true.

Step 330 processes $a = 2$ confronting $a \neq c$ and $b \neq a$ yielding two relevant clauses. One clause says $a = 2$ and $a \neq c$ imply $c \neq 2$. The other clause says more simply that $a = 2$ implies $b \neq 2$ (since we know $b \neq a$ form Step 15). (Recall we know $c \neq a$ is true from Step 61, but we have not determined $a \neq c$ via propositional clauses.)

Step 361 processes $c \neq a$ and is confronted by $c = 0$ and $c = 2$ yielding two relevant clauses. The content of these clauses are potentially confusing, so we explain in some depth. The first says if $c \neq a$ and $c = 0$, then we must either have $a \neq c$ or $c \neq c$. Since we know $c = c$ (from Step 62) we can simplify this to saying if $c \neq a$ and $c = 0$, then $a \neq c$. This is clearly true mathematically, as equality (and hence disequality) is symmetric, but is confusing since it does not really depend on the hypothesis $c = 0$. It becomes potentially more confusing once we realize we already know $c \neq a$ (from Step 61) and so the clause simplifies to saying $c = 0$ implies $a \neq c$. Technically we still do not have propositional clauses giving us $a \neq c$ (even though we do have $c \neq a$), but this conditional clause will allow us to infer $a \neq c$ when we are in the case that $c = 0$. The second clause is similar and after simplification says $c = 2$ implies $a \neq c$.

[1]It appears the long chain of events from Step 84 to Step 98, including the introduction of the eigenvariable e, was all in service of a reason to consider $d \neq d$ and determining $d = d$ is true. This could have happened in different (more reasonable) ways, e.g., by mating $d \in c$ and $d \notin b$, but this is how it happened in this particular case.

57: -20		-176	20 ∣ -176
58: -176		177 -178	176 ∣ 177
			176 ∣ -178
59: 177	Mating	-180 -185	
		-188	
60: -188	Confrontation	-192	
61: 15	Mating		-15 ∣ 19 ∣ -45 ∣ -185
		-204	
	Shallow Rule a7e5	207	-15 ∣ 207 ∣ -12
62: -204			204
65: 207	Shallow Rule b12c	214	-207 ∣ 214 ∣ -1
68: 214		219 -220	-214 ∣ 219
			-214 ∣ -220
69: 219	Shallow Rule 3de6	223	
70: -220	Shallow Rule 746f		220 ∣ -192 ∣ -4
74: 223		249 198	
75: 249		195	
76: 195			
82: 198			
84: -139	Shallow Rule cb84	-270	
89: -270		-285	
90: -285		286	
91: 286	Mating	-292	
92: -292	Shallow Rule a7b0	-301	
93: -301	Mating	-304	
96: -304	Shallow Rule a7b0	-311	
97: -311	Mating	-321	
98: -321			321

169: -150	Shallow Rule 746f	150 ∣ -113 ∣ -4
172: -23	Confrontation	23 ∣ -126 ∣ -21 ∣ -195
195: -113	Shallow Rule 63ab	113 ∣ -149 ∣ 126 ∣ 88 ∣ -6
202: -192	Shallow Rule 63ab	192 ∣ -219 ∣ 195 ∣ 198 ∣ -6
223: -180	Confrontation	180 ∣ -198 ∣ -204 ∣ -83
		180 ∣ -195 ∣ -204 ∣ -111
244: -110	Shallow Rule 63ab	110 ∣ -68 ∣ 111 ∣ 83 ∣ -6
330: 88	Confrontation	23 ∣ -88 ∣ -21 ∣ -198
		44 ∣ -88 ∣ -83 ∣ -21
361: -185	Confrontation	185 ∣ -198 ∣ -204 ∣ -23
		185 ∣ -195 ∣ -204 ∣ -23
369: -178	Mating	-177 ∣ 178 ∣ -180 ∣ -321

Figure 3.4: Search Steps Leading to Proof of Subgoal 2 of Theorem 3 (Part 2)

The final step, Step 369, processes $d \notin b$ mating it with $d \in c$ yielding a clause that finally gives $c \neq b$. Combining this with Step 223 we now know $c = 0$ implies $b \neq 0$ and $c = 2$ implies $b \neq 2$.

At this point MiniSat determines the clauses are unsatisfiable. Let us consider cases to determine this for ourselves. From Step 202 we know either $c = 0$ or $c = 2$ must be true.

Suppose $c = 0$ is true. From Step 361 we know $a \neq c$. By Step 195 either $a = 0$ or $a = 2$. If $a = 0$ were true, then Step 172 would imply $c \neq 0$, contradicting the assumption of this case. Hence we must have $a = 2$. By Step 244 $b = 0$ or $b = 2$. Step 369 contradicts $b = 0$. Step 330 contradicts $b = 2$.

Suppose $c = 2$ is true. From Step 361 we know $a \neq c$. By Step 195 either $a = 0$ or $a = 2$. If $a = 2$ were true, then Step 330 we would know $c \neq 2$, a contradiction. Hence we must have $a = 0$. By Step 244 $b = 0$ or $b = 2$. Step 47 contradicts $b = 0$. Step 369 contradicts $b = 2$.

Chapter 4

Theorem 10

Theorem 10: exactly2 $(1 \cup \{3\})$

4.1 Proof 1

4.1.1 Subgoal 1

4.1.2 Subgoal 2

14	$\exists x.(x \in (1 \cup \{3\}) \wedge \neg\forall y.(y \in (1 \cup \{3\}) \to y \in (\wp x)))$
15	$(4 \in (1 \cup \{3\}) \wedge \neg\forall x.(x \in (1 \cup \{3\}) \to x \in (\wp 4)))$
20	$(0 \in (1 \cup \{3\}) \wedge \neg\forall x.(x \in (1 \cup \{3\}) \to x \in (\wp 0)))$
28	$\forall x.(x \in (1 \cup \{3\}) \to x \in (\wp 0))$
29	$0 \in (1 \cup \{3\})$
31	$(\{3\} \in (1 \cup \{3\}) \to \{3\} \in (\wp 0))$
33	$(1 \in (1 \cup \{3\}) \to 1 \in (\wp 0))$
34	$(3 \in (1 \cup \{3\}) \to 3 \in (\wp 0))$
37	$\forall x.(x \in (1 \cup \{3\}) \to x \in (\wp 4))$
51	$\{3\} \in 4$
53	$(1 \cup \{3\}) \in 4$
60	$2 \in 0$
61	$2 \in 4$
67	$(0 \in 1 \vee 0 \in \{3\})$
72	$0 \in 0$
74	$(0 = 0)$
76	$1 \in 0$
80	$0 \in 1$
86	$(2 \in (1 \cup \{3\}) \to 2 \in (\wp 4))$
107	$(((((1 \cup \{3\}) = 0) \vee ((1 \cup \{3\}) = 1)) \vee ((1 \cup \{3\}) = 2)) \vee ((1 \cup \{3\}) = 3))$
110	$((((\{3\} = 0) \vee (\{3\} = 1)) \vee (\{3\} = 2)) \vee (\{3\} = 3))$
112	$(((\{3\} = 0) \vee (\{3\} = 1)) \vee (\{3\} = 2))$
122	$(\{3\} = 2)$
129	$3 \in \{3\}$
149	$((1 \cup \{3\}) = 3)$
151	$(1 \cup \{3\}) \in \{3\}$
157	$3 \in (1 \cup \{3\})$
170	$(4 = 2)$
195	$(3 = 3)$
199	$(2 = 2)$
200	$(2 = 1)$
245	$2 \in 3$
265	$((((2 = 0) \vee (2 = 1)) \vee (2 = 2)) \vee (2 = 3))$
266	$(((2 = 0) \vee (2 = 1)) \vee (2 = 2))$
267	$((2 = 0) \vee (2 = 1))$
299	$2 \in (\wp 4)$
331	$3 \in (\wp 0)$
333	$\forall x.(x \in 3 \to x \in 0)$
336	$(2 \in 3 \to 2 \in 0)$
341	$1 \in (\wp 0)$
343	$\forall x.(x \in 1 \to x \in 0)$
346	$(2 \in 1 \to 2 \in 0)$
347	$(1 \in 1 \to 1 \in 0)$
349	$(0 \in 1 \to 0 \in 0)$
361	$\{3\} \in (\wp 0)$
363	$\forall x.(x \in \{3\} \to x \in 0)$
368	$(3 \in \{3\} \to 3 \in 0)$
978	$(3 \in 1 \vee 3 \in \{3\})$

Table 4.1: Propositions in Subgoal 1 of Theorem 10 Search

59:	122			
61:	107		149	
62:	149	Shallow Rule 0e9c	151	
63:	151	Shallow Rule eef4	-157	
		Confrontation	-170	
			-195	
101:	-195			195
102:	-170	Confrontation	-199	
103:	-199	Shallow Rule eef4	-245	
		Shallow Rule f018	61	
155:	61	Shallow Rule 24be	265	
156:	265		266	
157:	266		267	
159:	267		200	
160:	200	Shallow Rule 73d1	245	
163:	245			
189:	86		299	
190:	299	Shallow Rule 70b2	202	
223:	34		331	
				-34 ‖ -157 ‖ 331
224:	331	Shallow Rule 70b2	333	
				-331 ‖ 333 ‖ -13
225:	333		336	
			Shallow Rule b187	
				-57 ‖ 72 ‖ -333
232:	336			
234:	33		341	
235:	341	Shallow Rule 70b2	343	
236:	343		346	
			347	
			349	
237:	346			
240:	349			
242:	347			
255:	31		361	
256:	361	Shallow Rule 70b2	363	
257:	363		368	
			Shallow Rule 0ed3	
259:	368	Shallow Rule 4fe3		57 ‖ -189 ‖ -6
				189 ‖ -190
				190 ‖ -74
644:	-245	Shallow Rule 4fe3	-266	
682:	-67			67 ‖ -80
723:	-157	Shallow Rule 2264	-978	
				157 ‖ -978 ‖ -3
724:	-978			978 ‖ -129
733:	-129	Shallow Rule 1314		129 ‖ -195 ‖ -4
		Shallow Rule 5fd2		129 ‖ -195 ‖ -4
741:	-80	Shallow Rule 35f1		80 ‖ -74 ‖ -8

Figure 4.1: Search Steps Leading to Proof of Subgoal 1 of Theorem 10

13	$a \in (1 \cup \{3\})$
14	$b \in Y$
15	$c \in Y$
18	$c \in (\wp b)$
19	$a \in Y$
20	$(c = a)$
21	$(Y = Y)$
24	$c \in (1 \cup \{3\})$
33	$\forall x.(x \in c \rightarrow x \in b)$
35	$(b = a)$
38	$b \in (1 \cup \{3\})$
44	$((1 \cup \{3\}) = Y)$
47	$(b \in 1 \vee b \in \{3\})$
68	$b \in \{3\}$
69	$b \in 1$
76	$(b = 3)$
80	$((1 \cup \{3\}) = 3)$
81	$(b = b)$
82	$(c = b)$
86	$b \in 4$
89	$(((b = 0) \vee (b = 1)) \vee (b = 2))$
90	$(b = 2)$
91	$((b = 0) \vee (b = 1))$
100	$b \in 3$
104	$\forall x.(x \in 2 \rightarrow x \in b)$
105	$(b = 1)$
106	$(b = 0)$
113	$((((b = 0) \vee (b = 1)) \vee (b = 2)) \vee (b = 3))$
122	$b \in 2$
141	$b \in \{0\}$
162	$(d \in c \rightarrow d \in b)$
163	$d \in c$
164	$d \in b$
167	$(d = d)$
202	$(c \in 1 \vee c \in \{3\})$
210	$c \in \{3\}$
211	$c \in 1$
222	$(a = a)$
225	$(a \in 1 \vee a \in \{3\})$
231	$(1 = a)$
232	$(\{3\} = a)$
351	$(a = \{0\})$
569	$(\{3\} = \{2\})$
572	$(d = \{2\})$
574	$(1 = \{2\})$
578	$(\{0\} = \{2\})$
593	$(\{2\} = 0)$
766	$(\{2\} = \{3\})$
827	$(4 = \{\{2\}\})$
986	$d \in \{\{2\}\}$
1499	$a \in \{3\}$
1500	$a \in 1$
5432	$((((1 \cup \{3\}) = 0) \vee ((1 \cup \{3\}) = 1)) \vee ((1 \cup \{3\}) = 2))$
5433	$(((1 \cup \{3\}) = 0) \vee ((1 \cup \{3\}) = 1))$
6083	$((1 = 0) \vee (1 = 1))$
6410	$(d \in 2 \rightarrow d \in b)$

Table 4.2: Propositions in Subgoal 2 of Theorem 10 Search

67:	13	Mating	-222	
		Shallow Rule 608e	225	
				-13 ∥ 225 ∥ -3
69:	-222	Mating	-231	
			-232	
			-236	
96:	0	Mating	-351	
99:	-351	Mating	-464	
		Confrontation	-546	
		Mating	-569	
			-572	
			-574	
			-577	
			-578	
		Confrontation	-593	
		Mating	-766	
		Mating	-827	
		Mating	-894	
220:	-572	Shallow Rule 69d3	-986	
317:	225		1500	
			1499	
				-225 ∥ 1500 ∥ 1499
358:	4			
563:	1499	Shallow Rule ec6b		-1499 ∥ 85 ∥ -4
564:	1500	Shallow Rule b66b		-1500 ∥ 140 ∥ -8
		Confrontation		31 ∥ -83 ∥ -85 ∥ -29
788:	2			
871:	-827			
900:	-766			
909:	-569			
910:	-578			
917:	-574			
972:	-232			
978:	-986			
1003:	{2}			
1005:	-82	Confrontation		82 ∥ -83 ∥ -29 ∥ -76
1039:	-80	Shallow Rule c40a	5432	
1040:	5432		5433	
1041:	5433			
1107:	-44			
1147:	{3}	Shallow Rule 2482	6083	
1256:	6083			
1278:	-35	Confrontation		35 ∥ -106 ∥ -81 ∥ -140
				35 ∥ -76 ∥ -81 ∥ -85
1279:	-593			
1349:	-104		-6410	
1350:	-6410			
1356:	-231	Confrontation		82 ∥ -139 ∥ -29 ∥ -106
				31 ∥ -139 ∥ -140 ∥ -29

Figure 4.2: Search Steps Leading to Proof of Subgoal 2 of Theorem 10

Bibliography

[1] Andrews, P.B., Brown, C.E.: TPS: A hybrid automatic-interactive system for developing proofs. Journal of Applied Logic 4(4), 367–395 (2006)

[2] Backes, J., Brown, C.E.: Analytic tableaux for higher-order logic with choice. Journal of Automated Reasoning 47(4), 451–479 (2011), dOI 10.1007/s10817-011-9233-2

[3] Biere, A.: Picosat essentials. Journal on Satisfiability, Boolean Modeling and Computation 4, 75–97 (2008)

[4] Brown, C.E.: Satallax: An automatic higher-order prover. In: Gramlich, B., Miller, D., Sattler, U. (eds.) IJCAR. LNCS, vol. 7364, pp. 111–117. Springer (2012)

[5] Brown, C.E.: Reducing higher-order theorem proving to a sequence of sat problems. Journal of Automated Reasoning 51(1), 57–77 (Mar 2013)

[6] Brown, C.E., Smolka, G.: Analytic tableaux for simple type theory and its first-order fragment. Logical Methods in Computer Science 6(2) (Jun 2010)

[7] Eén, N., Sörensson, N.: An extensible SAT-solver. In: Giunchiglia, E., Tacchella, A. (eds.) SAT. LNCS, vol. 2919, pp. 502–518. Springer (2003)

[8] Kovács, L., Voronkov, A.: First-order theorem proving and Vampire. In: Sharygina, N., Veith, H. (eds.) CAV. LNCS, vol. 8044, pp. 1–35. Springer (2013)

[9] McCune, W.: Prover9 and mace4 (2005–2010), `http://www.cs.unm.edu/~mccune/prover9/`

[10] Miller, D.A.: A compact representation of proofs. Studia Logica 46(4), 347–370 (1987)

[11] Schulz, S.: System Description: E 1.8. In: McMillan, K., Middeldorp, A., Voronkov, A. (eds.) Proc. of the 19th LPAR, Stellenbosch. LNCS, vol. 8312. Springer (2013)