

thm\_2EHolSmt\_2Er087  
(TMSj8BFXBijdue9Jgf4ahKGuMCEEyVa8cL4)

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**Definition 1** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p \text{ (ap } P \ x)) \text{ then (the } (\lambda x. x \in A \wedge p \ x)) \text{ of type } \iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj\_o } (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define `c_2Ebool_2E_3F` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } V0P \text{ (ap (c_2Emin_2E_40 } A \text{ (ap } P \ x))$

**Definition 4** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. \text{inj\_o } (p \Rightarrow q)$  of type  $\iota$ .

**Definition 5** We define `c_2Ebool_2E_T` to be  $(\text{ap (ap (c_2Emin_2E_3D } (2^2)) (\lambda V0x \in 2. V0x)) (\lambda V1x \in 2. V1x))$

**Definition 6** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap (ap (c_2Emin_2E_3D } (2^{A-27a})$

**Definition 7** We define `c_2Ebool_2E_5C_2F` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap (c_2Ebool_2E_21 } 2) (\lambda V2t \in 2. V2t))$

Let `ty_2Enum_2Enum` :  $\iota$  be given. Assume the following.

$$\text{nonempty } ty\_2Enum\_2Enum \tag{1}$$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \forall A1. \text{nonempty } A1 \Rightarrow \text{nonempty (ty\_2Epair\_2Eprod } A0 \ A1) \tag{2}$$

Let `ty_2Einteger_2Eint` :  $\iota$  be given. Assume the following.

$$\text{nonempty } ty\_2Einteger\_2Eint \tag{3}$$

Let `c_2Einteger_2Eint__REP__CLASS` :  $\iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod \ ty\_2Enum\_2Enum \ ty\_2Enum\_2Enum)}) ty\_2Einteger\_2Eint) \tag{4}$$

**Definition 8** We define `c_2Einteger_2Eint__REP` to be  $\lambda V0a \in ty\_2Einteger\_2Eint. (\text{ap (c_2Emin_2E_40 } (ty\_2Einteger\_2Eint\_REP\_CLASS \ a))$

Let  $c\_2Einteger\_2Etint\_mul : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum))\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum))\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum) \quad (5)$$

Let  $c\_2Einteger\_2Etint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)) \quad (6)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})} \quad (7)$$

**Definition 9** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)$

**Definition 10** We define  $c\_2Einteger\_2Eint\_mul$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint$

Let  $c\_2Einteger\_2Etint\_lt : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)) \quad (8)$$

**Definition 11** We define  $c\_2Einteger\_2Eint\_lt$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum)^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum} \quad (9)$$

**Definition 12** We define  $c\_2Enumeral\_2EiZ$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Einteger\_2Etint\_add : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum))\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum))\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum) \quad (10)$$

**Definition 13** We define  $c\_2Einteger\_2Eint\_add$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint$

Let  $c\_2Einteger\_2Etint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)) \quad (11)$$

**Definition 14** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap\ c\_2Einteger\_2Eint$

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint)^{ty\_2Enum\_2Enum} \quad (12)$$

**Definition 15** We define  $c\_Ebool\_EF$  to be  $(ap (c\_Ebool\_E21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 16** We define  $c\_Ebool\_E7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_Emin\_E3D\_3D\_3E V0t) c\_Ebool\_E21 2))$ .

**Definition 17** We define  $c\_Einteger\_Eint\_le$  to be  $\lambda V0x \in ty\_Einteger\_Eint.\lambda V1y \in ty\_Einteger\_Eint$ .

Let  $c\_Enum\_EZERO\_REP : \iota$  be given. Assume the following.

$$c\_Enum\_EZERO\_REP \in \omega \tag{13}$$

Let  $c\_Enum\_EABS\_num : \iota$  be given. Assume the following.

$$c\_Enum\_EABS\_num \in (ty\_Enum\_Enum^{\omega}) \tag{14}$$

**Definition 18** We define  $c\_Enum\_E0$  to be  $(ap c\_Enum\_EABS\_num c\_Enum\_EZERO\_REP)$ .

Let  $c\_Enum\_EREP\_num : \iota$  be given. Assume the following.

$$c\_Enum\_EREP\_num \in (\omega^{ty\_Enum\_Enum}) \tag{15}$$

Let  $c\_Enum\_ESUC\_REP : \iota$  be given. Assume the following.

$$c\_Enum\_ESUC\_REP \in (\omega^{\omega}) \tag{16}$$

**Definition 19** We define  $c\_Enum\_ESUC$  to be  $\lambda V0m \in ty\_Enum\_Enum.(ap c\_Enum\_EABS\_num c\_Enum\_ESUC\_REP m)$ .

**Definition 20** We define  $c\_Earithmetic\_EBIT2$  to be  $\lambda V0n \in ty\_Enum\_Enum.(ap (ap c\_Earithmetic\_EBIT2 c\_Enum\_E0) n)$ .

**Definition 21** We define  $c\_Earithmetic\_EBIT1$  to be  $\lambda V0n \in ty\_Enum\_Enum.(ap (ap c\_Earithmetic\_EBIT1 c\_Enum\_E0) n)$ .

**Definition 22** We define  $c\_Earithmetic\_EZERO$  to be  $c\_Enum\_E0$ .

**Definition 23** We define  $c\_Ebool\_E2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_Ebool\_E21 2) (\lambda V2t \in 2.V2t) t1 t2)))$ .

**Definition 24** We define  $c\_Eprim\_rec\_E3C$  to be  $\lambda V0m \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$ .

**Definition 25** We define  $c\_Earithmetic\_E3C\_3D$  to be  $\lambda V0m \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$ .

Let  $c\_Enumeral\_EiSUB : \iota$  be given. Assume the following.

$$c\_Enumeral\_EiSUB \in (((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum})^2) \tag{17}$$

**Definition 26** We define  $c\_Ebool\_ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t \in A\_27a.(ap c\_Ebool\_E2F\_5C t1 t2))))$ .

Let  $c\_Earithmetic\_E2D : \iota$  be given. Assume the following.

$$c\_Earithmetic\_E2D \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \tag{18}$$

**Definition 27** We define  $c\_Earithmetic\_ENUMERAL$  to be  $\lambda V0x \in ty\_Enum\_Enum.V0x$ .

Assume the following.

$$True \quad (19)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2.((\exists V1x \in A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (20)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \vee (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \vee True) \Leftrightarrow True) \wedge \\ & (((False \vee (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee False) \Leftrightarrow (p\ V0t)) \wedge ((p\ V0t) \vee \\ & (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \quad (21) \end{aligned}$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \quad (22) \end{aligned}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (23)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg(p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ & p\ V0t)))))) \quad (25) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t1 \in A\_27a.(\forall V1t2 \in \\ & A\_27a.(((ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ c\_2Ebool\_2ET)\ V0t1) \\ & V1t2) = V0t1) \wedge ((ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ c\_2Ebool\_2EF) \\ & V0t1)\ V1t2) = V1t2)))) \quad (26) \end{aligned}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\exists V1x \in A\_27a.(p\ (ap\ V0P\ V1x)))) \Leftrightarrow (\forall V2x \in A\_27a.(\neg(p\ (ap\ V0P\ V2x)))))) \quad (27)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p\ V0A) \wedge (p\ V1B))) \Leftrightarrow ((\neg( \\ & p\ V0A)) \vee (\neg(p\ V1B)))) \wedge ((\neg((p\ V0A) \vee (p\ V1B))) \Leftrightarrow ((\neg(p\ V0A)) \wedge (\neg(p\ V1B)))))) \quad (28) \end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((p (ap (ap (ap c\_2Einteger\_2Eint\_lt V0x) V1y)) \Leftrightarrow (p (ap (ap (ap c\_2Einteger\_2Eint\_le \\
& (ap (ap c\_2Einteger\_2Eint\_add V0x) (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO)))))) \\
& V1y))))))
\end{aligned} \tag{29}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((p (ap (ap c\_2Einteger\_2Eint\_le V0x) V1y)) \Leftrightarrow (p (ap (ap (ap c\_2Einteger\_2Eint\_le \\
& (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_add \\
& V1y) (ap c\_2Einteger\_2Eint\_neg V0x))))))
\end{aligned} \tag{30}$$

Assume the following.

$$\begin{aligned}
& (\forall V0y \in ty\_2Einteger\_2Eint. (\forall V1x \in ty\_2Einteger\_2Eint. \\
& ((ap (ap c\_2Einteger\_2Eint\_add V1x) V0y) = (ap (ap c\_2Einteger\_2Eint\_add \\
& V0y) V1x))))
\end{aligned} \tag{31}$$

Assume the following.

$$\begin{aligned}
& (\forall V0z \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_add \\
& V2x) (ap (ap c\_2Einteger\_2Eint\_add V1y) V0z)) = (ap (ap c\_2Einteger\_2Eint\_add \\
& (ap (ap c\_2Einteger\_2Eint\_add V2x) V1y)) V0z))))))
\end{aligned} \tag{32}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_add \\
& (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) V0x) = V0x))
\end{aligned} \tag{33}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_add \\
& V0x) (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) = V0x))
\end{aligned} \tag{34}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_mul \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO)))) V0x) = V0x))
\end{aligned} \tag{35}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2z \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_mul \\
& (ap (ap c\_2Einteger\_2Eint\_add V0x) V1y)) V2z) = (ap (ap c\_2Einteger\_2Eint\_add \\
& (ap (ap c\_2Einteger\_2Eint\_mul V0x) V2z)) (ap (ap c\_2Einteger\_2Eint\_mul \\
& V1y) V2z))))))
\end{aligned} \tag{36}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((ap\ c\_2Einteger\_2Eint\_neg\ (ap\ (ap\ c\_2Einteger\_2Eint\_add\ V0x) \\
& V1y)) = (ap\ (ap\ c\_2Einteger\_2Eint\_add\ (ap\ c\_2Einteger\_2Eint\_neg \\
& V0x))\ (ap\ c\_2Einteger\_2Eint\_neg\ V1y))))))
\end{aligned} \tag{37}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. ((ap\ (ap\ c\_2Einteger\_2Eint\_mul \\
& (ap\ c\_2Einteger\_2Eint\_of\_num\ c\_2Enum\_2E0))\ V0x) = (ap\ c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)))
\end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((ap\ c\_2Einteger\_2Eint\_neg\ (ap\ (ap\ c\_2Einteger\_2Eint\_mul\ V0x) \\
& V1y)) = (ap\ (ap\ c\_2Einteger\_2Eint\_mul\ (ap\ c\_2Einteger\_2Eint\_neg \\
& V0x))\ V1y))))))
\end{aligned} \tag{39}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((ap\ c\_2Einteger\_2Eint\_neg\ (ap\ (ap\ c\_2Einteger\_2Eint\_mul\ V0x) \\
& V1y)) = (ap\ (ap\ c\_2Einteger\_2Eint\_mul\ V0x)\ (ap\ c\_2Einteger\_2Eint\_neg \\
& V1y))))))
\end{aligned} \tag{40}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((\neg(p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le\ V0x)\ V1y))) \Leftrightarrow (p\ (ap\ (ap\ c\_2Einteger\_2Eint\_lt \\
& V1y)\ V0x))))))
\end{aligned} \tag{41}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (((p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le\ V0x)\ V1y)) \wedge (p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le \\
& V1y)\ V0x))) \Leftrightarrow (V0x = V1y))))))
\end{aligned} \tag{42}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1n \in ty\_2Enum\_2Enum. \\
& (\forall V2m \in ty\_2Enum\_2Enum. (((ap (ap c\_2Einteger\_2Eint\_add \\
& (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) V0p) = V0p) \wedge ((( \\
& ap (ap c\_2Einteger\_2Eint\_add V0p) (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) = V0p) \wedge (((ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) = (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) \wedge \\
& (((ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_neg V0p)) = \\
& V0p) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V1n))) (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V2m))) = (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Enumeral\_2EiZ (ap (ap c\_2Earithmetic\_2E\_2B \\
& V1n) V2m)))))) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V1n))) (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V2m)))) = (ap (ap (ap (c\_2Ebool\_2ECOND ty\_2Einteger\_2Eint) (ap \\
& (ap c\_2Earithmetic\_2E\_3C\_3D V2m) V1n)) (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V1n) \\
& V2m)))) (ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V2m) \\
& V1n)))))) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V1n))) (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V2m))) = (ap (ap (ap (c\_2Ebool\_2ECOND ty\_2Einteger\_2Eint) (ap ( \\
& ap c\_2Earithmetic\_2E\_3C\_3D V1n) V2m)) (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V2m) \\
& V1n)))) (ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V1n) \\
& V2m)))))) \wedge (((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& V1n))) (ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap c\_2Earithmetic\_2ENUMERAL V2m)))) = (ap c\_2Einteger\_2Eint\_neg \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmetic\_2ENUMERAL \\
& (ap c\_2Enumeral\_2EiZ (ap (ap c\_2Earithmetic\_2E\_2B V1n) V2m)))))))))))))) \\
& \hspace{15em} (43)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& \quad ((p (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow \\
& \quad True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad V0n)))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad (ap c\_2Arithmetic\_2ENUMERAL (ap c\_2Arithmetic\_2EBIT1 V0n)))))) \Leftrightarrow \\
& \quad False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad (ap c\_2Arithmetic\_2ENUMERAL (ap c\_2Arithmetic\_2EBIT2 V0n)))))) \Leftrightarrow \\
& \quad False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad (ap c\_2Arithmetic\_2ENUMERAL (ap c\_2Arithmetic\_2EBIT1 V0n)))) \\
& \quad \quad (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow False) \wedge (((p \\
& \quad \quad (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad (ap c\_2Arithmetic\_2ENUMERAL (ap c\_2Arithmetic\_2EBIT2 V0n)))) \\
& \quad \quad (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow False) \wedge (((p \\
& \quad \quad (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_neg (ap \\
& \quad \quad c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL (ap \\
& \quad \quad c\_2Arithmetic\_2EBIT1 V0n)))) (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad c\_2Enum\_2E0))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap \\
& \quad \quad c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad (ap c\_2Arithmetic\_2EBIT2 V0n)))) (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad c\_2Enum\_2E0))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap \\
& \quad \quad c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL V0n))) \\
& \quad \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad V1m)))) \Leftrightarrow (p (ap (ap c\_2Arithmetic\_2E\_3C\_3D V0n) V1m))) \wedge (((p ( \\
& \quad \quad ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad (ap c\_2Arithmetic\_2ENUMERAL V0n))) (ap c\_2Integer\_2Eint\_neg \\
& \quad \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad (ap c\_2Arithmetic\_2EBIT1 V1m)))))) \Leftrightarrow False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le \\
& \quad \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad V0n))) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad (ap c\_2Arithmetic\_2ENUMERAL (ap c\_2Arithmetic\_2EBIT2 V1m)))))) \Leftrightarrow \\
& \quad False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_neg \\
& \quad \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad V0n)))) (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad V1m)))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_le (ap c\_2Integer\_2Eint\_neg \\
& \quad \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Arithmetic\_2ENUMERAL \\
& \quad \quad V0n)))) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& \quad \quad (ap c\_2Arithmetic\_2ENUMERAL V1m)))))) \Leftrightarrow (p (ap (ap c\_2Arithmetic\_2E\_3C\_3D \\
& \quad \quad V1m) V0n)))))))))
\end{aligned}$$

(44)

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& ((p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Earithmic\_2EZERO) (ap c\_2Earithmic\_2EBIT1 \\
& V0n))) \Leftrightarrow True) \wedge (((p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Earithmic\_2EZERO) \\
& (ap c\_2Earithmic\_2EBIT2 V0n))) \Leftrightarrow True) \wedge (((p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& V0n) c\_2Earithmic\_2EZERO)) \Leftrightarrow False) \wedge (((p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& (ap c\_2Earithmic\_2EBIT1 V0n)) (ap c\_2Earithmic\_2EBIT1 V1m))) \Leftrightarrow \\
& (p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C V0n) V1m))) \wedge (((p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& (ap c\_2Earithmic\_2EBIT2 V0n)) (ap c\_2Earithmic\_2EBIT2 V1m))) \Leftrightarrow \\
& (p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C V0n) V1m))) \wedge (((p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& (ap c\_2Earithmic\_2EBIT1 V0n)) (ap c\_2Earithmic\_2EBIT2 V1m))) \Leftrightarrow \\
& (\neg (p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C V1m) V0n))) \wedge ((p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& (ap c\_2Earithmic\_2EBIT2 V0n)) (ap c\_2Earithmic\_2EBIT1 V1m))) \Leftrightarrow \\
& (p (ap (ap (ap c\_2Eprim\_rec\_2E\_3C V0n) V1m))))))))))))) \\
& \tag{45}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& ((p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D c\_2Earithmic\_2EZERO) V0n))) \Leftrightarrow \\
& True) \wedge (((p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D (ap c\_2Earithmic\_2EBIT1 \\
& V0n)) c\_2Earithmic\_2EZERO)) \Leftrightarrow False) \wedge (((p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D \\
& (ap c\_2Earithmic\_2EBIT2 V0n)) c\_2Earithmic\_2EZERO)) \Leftrightarrow False) \wedge \\
& (((p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D (ap c\_2Earithmic\_2EBIT1 \\
& V0n)) (ap c\_2Earithmic\_2EBIT1 V1m))) \Leftrightarrow (p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D \\
& V0n) V1m))) \wedge (((p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D (ap c\_2Earithmic\_2EBIT1 \\
& V0n)) (ap c\_2Earithmic\_2EBIT2 V1m))) \Leftrightarrow (p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D \\
& V0n) V1m))) \wedge (((p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D (ap c\_2Earithmic\_2EBIT2 \\
& V0n)) (ap c\_2Earithmic\_2EBIT1 V1m))) \Leftrightarrow (\neg (p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D \\
& V1m) V0n)))) \wedge (((p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D (ap c\_2Earithmic\_2EBIT2 \\
& V0n)) (ap c\_2Earithmic\_2EBIT2 V1m))) \Leftrightarrow (p (ap (ap (ap c\_2Earithmic\_2E\_3C\_3D \\
& V0n) V1m))))))))))))) \\
& \tag{46}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& (ap c\_2Earithmic\_2ENUMERAL (ap (ap (ap c\_2Earithmic\_2E\_2D V0n) \\
& V1m)) = (ap (ap (ap (c\_2Ebool\_2ECOND ty\_2Enum\_2Enum) (ap (ap (ap c\_2Eprim\_rec\_2E\_3C \\
& V1m) V0n)) (ap c\_2Earithmic\_2ENUMERAL (ap (ap (ap c\_2Enumeral\_2EiSUB \\
& c\_2Ebool\_2ET) V0n) V1m))) c\_2Enum\_2E0)))) \\
& \tag{47}
\end{aligned}$$

**Theorem 1**

$$\begin{aligned} & (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\ & (\forall V2z \in ty\_2Einteger\_2Eint. (\forall V3u \in ty\_2Einteger\_2Eint. \\ & ((ap (ap c\_2Einteger\_2Eint\_add V0x) (ap (ap c\_2Einteger\_2Eint\_add \\ V1y) (ap (ap c\_2Einteger\_2Eint\_add V2z) V3u))) = (ap (ap c\_2Einteger\_2Eint\_add \\ V1y) (ap (ap c\_2Einteger\_2Eint\_add V2z) (ap (ap c\_2Einteger\_2Eint\_add \\ V3u) V0x)))))))) \end{aligned}$$