

thm\_2EHolSmt\_2Er113  
(TMHmQ4rHEAv2sqHHhPi2VruxssXqZS4NvLu)

October 26, 2020

**Definition 1** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\mathbf{if} (\exists x \in A.p (ap P x)) \mathbf{then}$  (the  $(\lambda x.x \in A \wedge p x)$  of type  $\iota \Rightarrow \iota$ ).

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A a))))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{2}$$

Let  $ty\_2Einteger\_2Eint : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \tag{3}$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})_{ty\_2Einteger\_2Eint}) \tag{4}$$

**Definition 4** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 5** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a})) (ap V0P (ap (c\_2Emin\_2E\_40 A a))))))$

**Definition 6** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap (c\_2Emin\_2E\_40 (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)))$

Let  $c\_2Einteger\_2Eint\_It : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_It \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})_{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)}) \tag{5}$$

**Definition 7** We define  $c\_Einteger\_Eint\_lt$  to be  $\lambda V0T1 \in ty\_Einteger\_Eint.\lambda V1T2 \in ty\_Einteger\_Eint$ .

**Definition 8** We define  $c\_Ebool\_EF$  to be  $(ap (c\_Ebool\_E21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 9** We define  $c\_Emin\_E3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 10** We define  $c\_Ebool\_E7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_Emin\_E3D\_3D\_3E V0t) c\_Ebool\_E21 2))$ .

**Definition 11** We define  $c\_Einteger\_Eint\_le$  to be  $\lambda V0x \in ty\_Einteger\_Eint.\lambda V1y \in ty\_Einteger\_Eint$ .

**Definition 12** We define  $c\_Einteger\_Eint\_ge$  to be  $\lambda V0x \in ty\_Einteger\_Eint.\lambda V1y \in ty\_Einteger\_Eint$ .

Let  $c\_EEnum\_EZERO\_REP : \iota$  be given. Assume the following.

$$c\_EEnum\_EZERO\_REP \in \omega \tag{6}$$

Let  $c\_EEnum\_EABS\_num : \iota$  be given. Assume the following.

$$c\_EEnum\_EABS\_num \in (ty\_EEnum\_EEnum^{\omega}) \tag{7}$$

**Definition 13** We define  $c\_EEnum\_E0$  to be  $(ap c\_EEnum\_EABS\_num c\_EEnum\_EZERO\_REP)$ .

**Definition 14** We define  $c\_Earithmic\_EZERO$  to be  $c\_EEnum\_E0$ .

Let  $c\_Einteger\_Eint\_mul : \iota$  be given. Assume the following.

$$c\_Einteger\_Eint\_mul \in (((ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)_{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)})_{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)})_{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)} \tag{8}$$

Let  $c\_Einteger\_Eint\_eq : \iota$  be given. Assume the following.

$$c\_Einteger\_Eint\_eq \in ((2^{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)})_{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)})_{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)} \tag{9}$$

Let  $c\_Einteger\_Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_Einteger\_Eint\_ABS\_CLASS \in (ty\_Einteger\_Eint)^{(2^{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)})_{(ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)}} \tag{10}$$

**Definition 15** We define  $c\_Einteger\_Eint\_ABS$  to be  $\lambda V0r \in (ty\_Epair\_Eprod ty\_EEnum\_EEnum ty\_EEnum\_EEnum)$ .

**Definition 16** We define  $c\_Einteger\_Eint\_mul$  to be  $\lambda V0T1 \in ty\_Einteger\_Eint.\lambda V1T2 \in ty\_Einteger\_Eint$ .

Let  $c\_Earithmic\_E2D : \iota$  be given. Assume the following.

$$c\_Earithmic\_E2D \in ((ty\_EEnum\_EEnum^{ty\_EEnum\_EEnum})_{ty\_EEnum\_EEnum})_{ty\_EEnum\_EEnum} \tag{11}$$

**Definition 17** We define  $c\_Ebool\_E2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_Ebool\_E21 2) (\lambda V2t \in 2.V2t))))$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (12)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (13)$$

**Definition 18** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num$

**Definition 19** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 20** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

**Definition 21** We define  $c\_2Earithmetic\_2E\_3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 22** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (14)$$

**Definition 23** We define  $c\_2Enumeral\_2EiZ$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Einteger\_2Etint\_add : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (15)$$

**Definition 24** We define  $c\_2Einteger\_2Eint\_add$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint$

Let  $c\_2Einteger\_2Etint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (16)$$

**Definition 25** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap\ c\_2Einteger\_2Eint$

**Definition 26** We define  $c\_2Earithmetic\_2EBIT2$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

**Definition 27** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

**Definition 28** We define  $c\_2Earithmetic\_2ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (17)$$

Assume the following.

$$True \quad (18)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. ((p V0t1) \Rightarrow (p V1t2)) \Rightarrow ((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))) \quad (19)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (20)$$

Assume the following.

$$(\forall V0t \in 2. ((p V0t) \vee (\neg(p V0t)))) \quad (21)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2. ((\exists V1x \in A\_27a. (p V0t)) \Leftrightarrow (p V0t))) \quad (22)$$

Assume the following.

$$(\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))) \quad (23)$$

Assume the following.

$$(\forall V0t \in 2. (((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee (p V0t)) \Leftrightarrow (p V0t)))) \quad (24)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (25)$$

Assume the following.

$$((\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (26)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (27)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0x \in A.27a. (\forall V1y \in A.27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (28)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow \neg(p V0t)) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow \neg(p V0t)))))) \quad (29)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). (\neg(\exists V1x \in A.27a. (p (ap V0P V1x)))) \Leftrightarrow (\forall V2x \in A.27a. (\neg(p (ap V0P V2x)))))) \quad (30)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_lt V0x) V1y)) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_le (ap (ap c\_2Einteger\_2Eint\_add V0x) (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL (ap c\_2Earithmic\_2EBIT1 c\_2Earithmic\_2EZERO)))))) V1y)))))) \quad (31)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_le V0x) V1y)) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_le (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_add V1y) (ap c\_2Einteger\_2Eint\_neg V0x)))))) \quad (32)$$

Assume the following.

$$(\forall V0c \in ty\_2Einteger\_2Eint. (\forall V1x \in ty\_2Einteger\_2Eint. (\forall V2y \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_le (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_add V0c) V1x))) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_lt V2y) (ap c\_2Einteger\_2Eint\_neg V1x))) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_le (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_neg V0c) V2y))) \Leftrightarrow False)))))) \quad (33)$$

Assume the following.

$$(\forall V0y \in ty\_2Einteger\_2Eint. (\forall V1x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_add V1x) V0y) = (ap (ap c\_2Einteger\_2Eint\_add V0y) V1x)))) \quad (34)$$

Assume the following.

$$\begin{aligned}
& (\forall V0z \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_add \\
V2x) (ap (ap c\_2Einteger\_2Eint\_add V1y) V0z)) = (ap (ap c\_2Einteger\_2Eint\_add \\
& (ap (ap c\_2Einteger\_2Eint\_add V2x) V1y)) V0z))))))
\end{aligned} \tag{35}$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_add V0x) (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) = V0x)) \tag{36}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_mul \\
& (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& (ap c\_2Earithmic\_2EBIT1 c\_2Earithmic\_2EZERO)))) V0x) = V0x))
\end{aligned} \tag{37}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((ap c\_2Einteger\_2Eint\_neg (ap (ap c\_2Einteger\_2Eint\_add V0x) \\
V1y)) = (ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_neg \\
& V0x)) (ap c\_2Einteger\_2Eint\_neg V1y))))))
\end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((ap c\_2Einteger\_2Eint\_neg (ap (ap c\_2Einteger\_2Eint\_mul V0x) \\
V1y)) = (ap (ap c\_2Einteger\_2Eint\_mul (ap c\_2Einteger\_2Eint\_neg \\
& V0x)) V1y))))))
\end{aligned} \tag{39}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((ap c\_2Einteger\_2Eint\_neg (ap (ap c\_2Einteger\_2Eint\_mul V0x) \\
V1y)) = (ap (ap c\_2Einteger\_2Eint\_mul V0x) (ap c\_2Einteger\_2Eint\_neg \\
& V1y))))))
\end{aligned} \tag{40}$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint. ((ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_neg V0x)) = V0x)) \tag{41}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((\neg (p (ap (ap c\_2Einteger\_2Eint\_le V0x) V1y))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_lt \\
& V1y) V0x))))))
\end{aligned} \tag{42}$$

Assume the following.

$$((ap\ c\_2Einteger\_2Eint\_neg\ (ap\ c\_2Einteger\_2Eint\_of\_num\ c\_2Enum\_2E0)) = (ap\ c\_2Einteger\_2Eint\_of\_num\ c\_2Enum\_2E0)) \quad (43)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1n \in ty\_2Enum\_2Enum. \\ & (\forall V2m \in ty\_2Enum\_2Enum. (((ap\ (ap\ c\_2Einteger\_2Eint\_add \\ & (ap\ c\_2Einteger\_2Eint\_of\_num\ c\_2Enum\_2E0))\ V0p) = V0p) \wedge ((( \\ & ap\ (ap\ c\_2Einteger\_2Eint\_add\ V0p)\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) = V0p) \wedge (((ap\ c\_2Einteger\_2Eint\_neg\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) = (ap\ c\_2Einteger\_2Eint\_of\_num\ c\_2Enum\_2E0)) \wedge \\ & (((ap\ c\_2Einteger\_2Eint\_neg\ (ap\ c\_2Einteger\_2Eint\_neg\ V0p)) = \\ & V0p) \wedge (((ap\ (ap\ c\_2Einteger\_2Eint\_add\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ V1n)))\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ V2m))) = (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ (ap\ c\_2Enumeral\_2EiZ\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B \\ & V1n)\ V2m)))))) \wedge (((ap\ (ap\ c\_2Einteger\_2Eint\_add\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ V1n)))\ (ap\ c\_2Einteger\_2Eint\_neg \\ & (ap\ c\_2Einteger\_2Eint\_of\_num\ (ap\ c\_2Earithmetic\_2ENUMERAL \\ & V2m)))) = (ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ ty\_2Einteger\_2Eint)\ (ap \\ & (ap\ c\_2Earithmetic\_2E\_3C\_3D\ V2m)\ V1n))\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ (ap\ (ap\ c\_2Earithmetic\_2E\_2D\ V1n) \\ & V2m))))\ (ap\ c\_2Einteger\_2Eint\_neg\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ (ap\ (ap\ c\_2Earithmetic\_2E\_2D\ V2m) \\ & V1n)))))) \wedge (((ap\ (ap\ c\_2Einteger\_2Eint\_add\ (ap\ c\_2Einteger\_2Eint\_neg \\ & (ap\ c\_2Einteger\_2Eint\_of\_num\ (ap\ c\_2Earithmetic\_2ENUMERAL \\ & V1n)))\ (ap\ c\_2Einteger\_2Eint\_of\_num\ (ap\ c\_2Earithmetic\_2ENUMERAL \\ & V2m))) = (ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ ty\_2Einteger\_2Eint)\ (ap\ ( \\ & ap\ c\_2Earithmetic\_2E\_3C\_3D\ V1n)\ V2m))\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ (ap\ (ap\ c\_2Earithmetic\_2E\_2D\ V2m) \\ & V1n))))\ (ap\ c\_2Einteger\_2Eint\_neg\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ (ap\ (ap\ c\_2Earithmetic\_2E\_2D\ V1n) \\ & V2m)))))) \wedge (((ap\ (ap\ c\_2Einteger\_2Eint\_add\ (ap\ c\_2Einteger\_2Eint\_neg \\ & (ap\ c\_2Einteger\_2Eint\_of\_num\ (ap\ c\_2Earithmetic\_2ENUMERAL \\ & V1n)))\ (ap\ c\_2Einteger\_2Eint\_neg\ (ap\ c\_2Einteger\_2Eint\_of\_num \\ & (ap\ c\_2Earithmetic\_2ENUMERAL\ V2m)))) = (ap\ c\_2Einteger\_2Eint\_neg \\ & (ap\ c\_2Einteger\_2Eint\_of\_num\ (ap\ c\_2Earithmetic\_2ENUMERAL \\ & (ap\ c\_2Enumeral\_2EiZ\ (ap\ (ap\ c\_2Earithmetic\_2E\_2B\ V1n)\ V2m))))))))))))) \end{aligned} \quad (44)$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& \quad ((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_of\_num \\
& \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad (ap c\_2Earithmic\_2EBIT1 V0n)))))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt \\
& \quad (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num \\
& \quad (ap c\_2Earithmic\_2ENUMERAL (ap c\_2Earithmic\_2EBIT2 V0n)))))) \Leftrightarrow \\
& \quad True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_of\_num \\
& \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow \\
& \quad False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_of\_num \\
& \quad c\_2Enum\_2E0)) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& \quad (ap c\_2Earithmic\_2ENUMERAL V0n)))))) \Leftrightarrow False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt \\
& \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad V0n)) (ap c\_2Integer\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow False) \wedge \\
& \quad (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_neg \\
& \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad (ap c\_2Earithmic\_2EBIT1 V0n)))))) (ap c\_2Integer\_2Eint\_of\_num \\
& \quad c\_2Enum\_2E0))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap \\
& \quad c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad (ap c\_2Earithmic\_2EBIT2 V0n)))))) (ap c\_2Integer\_2Eint\_of\_num \\
& \quad c\_2Enum\_2E0))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap \\
& \quad c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL V0n))) \\
& \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad V1m)))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C V0n V1m))) \wedge (((p (ap (ap \\
& \quad c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& \quad (ap c\_2Earithmic\_2ENUMERAL (ap c\_2Earithmic\_2EBIT1 V0n)))))) \\
& \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad V1m)))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt (ap c\_2Integer\_2Eint\_neg \\
& \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad (ap c\_2Earithmic\_2EBIT2 V0n)))))) (ap c\_2Integer\_2Eint\_of\_num \\
& \quad (ap c\_2Earithmic\_2ENUMERAL V1m)))) \Leftrightarrow True) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt \\
& \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad V0n)) (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num \\
& \quad (ap c\_2Earithmic\_2ENUMERAL V1m)))))) \Leftrightarrow False) \wedge (((p (ap (ap c\_2Integer\_2Eint\_lt \\
& \quad (ap c\_2Integer\_2Eint\_neg (ap c\_2Integer\_2Eint\_of\_num ( \\
& \quad ap c\_2Earithmic\_2ENUMERAL V0n)))) (ap c\_2Integer\_2Eint\_neg \\
& \quad (ap c\_2Integer\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL \\
& \quad V1m)))))) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C V1m) V0n))))))))))))) \\
& \hspace{15em} (45)
\end{aligned}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (46)$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (47)$$



Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (48)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg(\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (49)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (50)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (p V1q) \Leftrightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg(p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \quad (51)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (p V1q) \wedge (p V2r)) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p)))))))) \quad (52)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (p V1q) \vee (p V2r)) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))) \quad (53)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p)))))) \quad (54)$$

### Theorem 1

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.(\forall V2z \in ty\_2Einteger\_2Eint.((p (ap (ap c\_2Einteger\_2Eint\_le V0x) (ap (ap c\_2Einteger\_2Eint\_add V1y) V2z))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_ge (ap (ap c\_2Einteger\_2Eint\_add V2z) (ap (ap c\_2Einteger\_2Eint\_mul (ap c\_2Einteger\_2Eint\_neg (ap c\_2Einteger\_2Eint\_of\_num (ap c\_2Earithmic\_2ENUMERAL (ap c\_2Earithmic\_2EBIT1 c\_2Earithmic\_2EZERO)))))) V0x))) (ap c\_2Einteger\_2Eint\_neg V1y))))))$$