

thm\_2Earithmetic\_2EMIN\_\_MAX\_\_LE (TMcG-BzVCgyTdmqN8kECkFZuiZFY5j7NVHHf)

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Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (1)$$

**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota. (\lambda V0P \in (2^{A\_27a}). (ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})) (\lambda V1P \in 2.V1P)) (\lambda V2P \in 2.V2P)))$

**Definition 4** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p \ P \Rightarrow p \ Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2. (ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2EF))$

**Definition 7** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2. inj\_o (V0t1 = V1t2))))$

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (2)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (3)$$

Let  $c\_2Enum\_2EAABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EAABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (4)$$

**Definition 8** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. (ap c\_2Enum\_2EAABS\_num m)$

**Definition 9** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p (ap P x)) \text{ then } (\text{the } (\lambda x. x \in A \wedge p$  of type  $\iota \Rightarrow \iota$ .

**Definition 10** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40$

**Definition 11** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.$

**Definition 12** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

**Definition 13** We define  $c\_2Earithmetic\_2E\_3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.$

**Definition 14** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.(\$

**Definition 15** We define  $c\_2Earithmetic\_2EMAX$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.$

**Definition 16** We define  $c\_2Earithmetic\_2EMIN$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.(\neg(p\ (ap\ (ap\ c\_2Eprim\_rec\_2E\_3C\ V0m)\ V1n)) \Leftrightarrow (p\ (ap\ (ap\ c\_2Earithmetic\_2E\_3C\_3D\ V1n)\ V0m)))))) \quad (5)$$

Assume the following.

$$True \quad (6)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2))))) \quad (7)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p\ V0t1) \wedge ((p\ V1t2) \wedge (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \wedge (p\ V2t3)))))) \quad (8)$$

Assume the following.

$$(\forall V0t \in 2.(((p\ V0t) \Rightarrow False) \Rightarrow (\neg(p\ V0t)))) \quad (9)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p\ V0t)) \Rightarrow ((p\ V0t) \Rightarrow False))) \quad (10)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg(p\ V0t))))))) \quad (11)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p\ V0t)) \Leftrightarrow (p\ V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True)))) \quad (12)$$

Assume the following.

$$\forall A_{\text{27a}}. \text{nonempty } A_{\text{27a}} \Rightarrow (\forall V0x \in A_{\text{27a}}. (\forall V1y \in A_{\text{27a}}. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (13)$$

Assume the following.

$$(\forall V0t \in 2. (((\text{True} \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow \text{True}) \Leftrightarrow (p V0t)) \wedge (((\text{False} \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow \text{False}) \Leftrightarrow (\neg(p V0t))))))) \quad (14)$$

Assume the following.

$$\forall A_{\text{27a}}. \text{nonempty } A_{\text{27a}} \Rightarrow (\forall V0P \in (2^{A_{\text{27a}}}). ((\neg(\forall V1x \in A_{\text{27a}}. (p (ap V0P V1x)))) \Leftrightarrow (\exists V2x \in A_{\text{27a}}. (\neg(p (ap V0P V2x))))))) \quad (15)$$

Assume the following.

$$\forall A_{\text{27a}}. \text{nonempty } A_{\text{27a}} \Rightarrow (\forall V0P \in (2^{A_{\text{27a}}}). (\forall V1Q \in (2^{A_{\text{27a}}}). ((\forall V2x \in A_{\text{27a}}. ((p (ap V0P V2x)) \wedge (p (ap V1Q V2x)))) \Leftrightarrow ((\forall V3x \in A_{\text{27a}}. (p (ap V0P V3x))) \wedge (\forall V4x \in A_{\text{27a}}. (p (ap V1Q V4x)))))))))) \quad (16)$$

Assume the following.

$$\forall A_{\text{27a}}. \text{nonempty } A_{\text{27a}} \Rightarrow (\forall V0P \in (2^{A_{\text{27a}}}). (\forall V1Q \in (2^{A_{\text{27a}}}). ((\exists V2x \in A_{\text{27a}}. ((p (ap V0P V2x)) \vee (p (ap V1Q V2x)))) \Leftrightarrow ((\exists V3x \in A_{\text{27a}}. (p (ap V0P V3x))) \vee (\exists V4x \in A_{\text{27a}}. (p (ap V1Q V4x)))))))))) \quad (17)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((p V0A) \vee (p V1B)) \Leftrightarrow ((p V1B) \vee (p V0A)))))) \quad (18)$$

Assume the following.

$$\begin{aligned} \forall A_{\text{27a}}. \text{nonempty } A_{\text{27a}} \Rightarrow & \forall A_{\text{27b}}. \text{nonempty } A_{\text{27b}} \Rightarrow \\ & (\forall V0b \in 2. (\forall V1f \in (A_{\text{27b}})^{A_{\text{27a}}}). (\forall V2g \in (A_{\text{27b}})^{A_{\text{27a}}}). \\ & (\forall V3x \in A_{\text{27a}}. ((ap (ap (ap (ap (c_2Ebool\_2ECOND (A_{\text{27b}})^{A_{\text{27a}}}) \\ & V0b) V1f) V2g) V3x) = (ap (ap (ap (c_2Ebool\_2ECOND A_{\text{27b}}) V0b) (ap \\ & V1f V3x)) (ap V2g V3x))))))) \end{aligned} \quad (19)$$

Assume the following.

$$\begin{aligned} \forall A_{\text{27a}}. \text{nonempty } A_{\text{27a}} \Rightarrow & \forall A_{\text{27b}}. \text{nonempty } A_{\text{27b}} \Rightarrow \\ & (\forall V0f \in (A_{\text{27b}})^{A_{\text{27a}}}). (\forall V1b \in 2. (\forall V2x \in A_{\text{27a}}. \\ & (\forall V3y \in A_{\text{27a}}. ((ap V0f (ap (ap (ap (c_2Ebool\_2ECOND A_{\text{27a}}) \\ & V1b) V2x) V3y) = (ap (ap (ap (c_2Ebool\_2ECOND A_{\text{27b}}) V1b) (ap V0f \\ & V2x)) (ap V0f V3y))))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} & (\forall V0b \in 2. (\forall V1t1 \in 2. (\forall V2t2 \in 2. ((p \text{ (ap (ap } \\ & \text{ (ap (c\_2Ebool\_2ECOND 2) V0b) V1t1) V2t2)) \Leftrightarrow (((\neg(p V0b)) \vee (p V1t1)) \wedge \\ & \text{ ((p V0b) \vee (p V2t2))))))) \end{aligned} \quad (21)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (22)$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (23)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\ & \text{ ((}(p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\ & \text{ ((}(p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \end{aligned} \quad (25)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\ & \text{ ((p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg(p V2r)) \vee ((\neg(p V1q)) \vee ((\neg(p V0p)) \vee ((\neg(p V1q) \vee ((\neg(p V0p)))))))))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\ & \text{ ((p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee ((\neg(p V2r)))) \wedge (((p V1q) \vee ((\neg(p V0p)) \wedge ((p V2r) \vee ((\neg(p V0p))))))))))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\ & \text{ ((p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \wedge (((p V0p) \vee ((\neg(p V2r)))) \wedge \\ & \text{ ((p V1q) \vee ((p V2r) \vee ((\neg(p V0p))))))))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\ & \text{ ((p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee ((\neg(p V2r)))) \wedge ((\neg(p V1q)) \vee ((p V2r) \vee ((\neg(p V0p))))))))))) \end{aligned} \quad (30)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (((p \ V0p) \Leftrightarrow (\neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))) \quad (31)$$

**Theorem 1**

$$(\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. (p \ (ap \ (ap \ c\_2Earithmetic\_2E\_3C\_3D \ (ap \ (ap \ c\_2Earithmetic\_2EMIN \ V0m) \ V1n)) \ (ap \ (ap \ c\_2Earithmetic\_2EMAX \ V0m) \ V1n))))))$$