

thm_2Ebag_2EBAG__GEN__PROD__UNION__LEM
 (TMJ7pjMRZrWBSVctCLHqh3TtXdCLnpUR3Vr)

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Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \quad (1)$$

Let $c_2Earithmetic_2E_2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2B \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (2)$$

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A. \lambda x \in A. \lambda y \in A. inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define c_2Ebool_2ET to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota. (\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a})) (\lambda V1P \in 2.V1P)) (\lambda V2P \in 2.V2P)))$

Definition 4 We define $c_2Ebag_2EBAG__UNION$ to be $\lambda A_27a : \iota. \lambda V0b \in (ty_2Enum_2Enum^{A_27a}). \lambda V1c \in (ty_2Enum_2Enum^{A_27a})$

Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in omega \quad (3)$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{omega}) \quad (4)$$

Definition 5 We define c_2Enum_2E0 to be $(ap c_2Enum_2EABS_num c_2Enum_2EZERO_REP)$.

Definition 6 We define $c_2Ecombin_2EK$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. (\lambda V0x \in A_27a. (\lambda V1y \in A_27b. V0x))$

Definition 7 We define $c_2Ebag_2EEMPTY_BAG$ to be $\lambda A_27a : \iota. (ap (c_2Ecombin_2EK ty_2Enum_2Enum^{A_27a}))$

Let $c_2Earithmetic_2E_2A : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2A \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (5)$$

Definition 8 We define $c_2Earithmetic_2EZERO$ to be c_2Enum_2E0 .

Let $c_2Enum_2EREPE_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (\omega^{ty_2Enum_2Enum}) \quad (6)$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (\omega^\omega) \quad (7)$$

Definition 9 We define c_2Enum_2ESUC to be $\lambda V0m \in ty_2Enum_2Enum.(ap\ c_2Enum_2EABS_num\ ($

Definition 10 We define $c_2Earthmetic_2EBIT1$ to be $\lambda V0n \in ty_2Enum_2Enum.(ap\ (ap\ c_2Earthmetic\ 0\ n)\ V)$

Definition 11 We define `c_2Earithmetic_2ENUMERAL` to be $\lambda V0x \in ty_2Enum_2Enum. V0x$.

Definition 12 We define c_2Ebool_2EEF to be $(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 13 We define $c_{\text{min}}(P)$ to be $\lambda P \in 2.\lambda Q \in 2.\text{inj_o}(p \ P \Rightarrow p \ Q)$ of type ι .

Definition 14 We define $c_{\text{C_Ebool_2E_2F_5C}}$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_{\text{C_Ebool_2E_21}} 2)) (\lambda V2t \in$

Definition 15 We define $c_{\text{Emin}}.40$ to be $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p \text{ (ap } P \text{ } x)) \text{ then } (\text{the } (\lambda x.x \in A \wedge p \text{ of type } \iota \Rightarrow \iota.$

Definition 16 We define c_Ebool_ECOND to be $\lambda A.27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A.27a.(\lambda V2t2 \in A.27a.($

Definition 17 We define $c_2EBag_EBAG_INSERT$ to be $\lambda A_27a : \iota. \lambda V0e \in A_27a. \lambda V1b \in (ty_2Enum_2E$

Let $c : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.\text{nonempty } A_27a \Rightarrow \forall A_27b.\text{nonempty } A_27b \Rightarrow c.2Ebag_2EITBAG \\ A_27a \ A_27b \in (((A_27b^A_27b)^{(ty_2Enum_2Enum^A_27a)})^{((A_27b^A_27b)^A_27a)}) \quad (8)$$

Definition 18 We define $c_2EBag_2EBAG_GEN_PROD$ to be $\lambda V0bag \in (ty_2Enum_2Enum\ ty_2Enum_2Enum)$

Definition 19 We define $c_2Ebag_2EFINITE_BAG$ to be $\lambda A.\lambda 27a : \iota.\lambda V0b \in (ty_2Enum_2Enum^A_{27a}).(ap$

Definition 20 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in$

Definition 21 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap\ (ap\ c_2Emin_2E_3D_3D_3E\ V0t)\ c_2Ebool_2E_7E))$

Assume the following.

($\forall V \exists m \in t$

$$(ap (ap c_ZArithmetc_ZE_ZA V0m) V1n) = (ap (ap c_ZArithmetc_ZE_ZA V1n) V0m))) \quad (9)$$

Assume the following.

$$\begin{aligned}
 & (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\forall V2p \in ty_2Enum_2Enum. ((ap (ap c_2Earithmetic_2E_2A V0m) \\
 & (ap (ap c_2Earithmetic_2E_2A V1n) V2p)) = (ap (ap c_2Earithmetic_2E_2A \\
 & (ap (ap c_2Earithmetic_2E_2A V0m) V1n)) V2p)))))) \\
 \end{aligned} \tag{10}$$

Assume the following.

$$\begin{aligned}
 & \forall A_27a.nonempty A_27a \Rightarrow (\forall V0e \in A_27a. (\forall V1b1 \in \\
 & (ty_2Enum_2Enum^{A_27a}). (\forall V2b2 \in (ty_2Enum_2Enum^{A_27a}). \\
 & ((ap (ap (c_2Ebag_2EBAG_UNION A_27a) (ap (ap (c_2Ebag_2EBAG_INSERT \\
 & A_27a) V0e) V1b1)) V2b2) = (ap (ap (c_2Ebag_2EBAG_INSERT A_27a) \\
 & V0e) (ap (ap (c_2Ebag_2EBAG_UNION A_27a) V1b1) V2b2))) \wedge ((ap (\\
 & ap (c_2Ebag_2EBAG_UNION A_27a) V1b1) (ap (ap (c_2Ebag_2EBAG_INSERT \\
 & A_27a) V0e) V2b2)) = (ap (ap (c_2Ebag_2EBAG_INSERT A_27a) V0e) \\
 & (ap (ap (c_2Ebag_2EBAG_UNION A_27a) V1b1) V2b2))))))) \\
 \end{aligned} \tag{11}$$

Assume the following.

$$\begin{aligned}
 & \forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow \forall A_27c. \\
 & nonempty A_27c \Rightarrow ((\forall V0b \in (ty_2Enum_2Enum^{A_27a}). ((ap (ap \\
 & (c_2Ebag_2EBAG_UNION A_27a) V0b) (c_2Ebag_2EEMPTY_BAG A_27a)) = \\
 & V0b)) \wedge ((\forall V1b \in (ty_2Enum_2Enum^{A_27b}). ((ap (ap (c_2Ebag_2EBAG_UNION \\
 & A_27b) (c_2Ebag_2EEMPTY_BAG A_27b) V1b) = V1b)) \wedge (\forall V2b1 \in \\
 & (ty_2Enum_2Enum^{A_27c}). (\forall V3b2 \in (ty_2Enum_2Enum^{A_27c}). \\
 & (((ap (ap (c_2Ebag_2EBAG_UNION A_27c) V2b1) V3b2) = (c_2Ebag_2EEMPTY_BAG \\
 & A_27c)) \Leftrightarrow ((V2b1 = (c_2Ebag_2EEMPTY_BAG A_27c)) \wedge (V3b2 = (c_2Ebag_2EEMPTY_BAG \\
 & A_27c)))))))))) \\
 \end{aligned} \tag{12}$$

Assume the following.

$$\begin{aligned}
 & \forall A_27a.nonempty A_27a \Rightarrow (\forall V0P \in (2^{(ty_2Enum_2Enum^{A_27a})}). \\
 & ((p (ap V0P (c_2Ebag_2EEMPTY_BAG A_27a))) \wedge (\forall V1b \in (ty_2Enum_2Enum^{A_27a}). \\
 & (((p (ap (c_2Ebag_2EFINITE_BAG A_27a) V1b)) \wedge (p (ap V0P V1b))) \Rightarrow \\
 & (\forall V2e \in A_27a. (p (ap V0P (ap (ap (c_2Ebag_2EBAG_INSERT A_27a) \\
 & V2e) V1b)))))) \Rightarrow (\forall V3b \in (ty_2Enum_2Enum^{A_27a}). ((p (ap \\
 & (c_2Ebag_2EFINITE_BAG A_27a) V3b)) \Rightarrow (p (ap V0P V3b)))))) \\
 \end{aligned} \tag{13}$$

Assume the following.

$$\begin{aligned}
 & \forall A_27a.nonempty A_27a \Rightarrow (\forall V0b1 \in (ty_2Enum_2Enum^{A_27a}). \\
 & (\forall V1b2 \in (ty_2Enum_2Enum^{A_27a}). ((p (ap (c_2Ebag_2EFINITE_BAG \\
 & A_27a) (ap (ap (c_2Ebag_2EBAG_UNION A_27a) V0b1) V1b2))) \Leftrightarrow ((p \\
 & (ap (c_2Ebag_2EFINITE_BAG A_27a) V0b1)) \wedge (p (ap (c_2Ebag_2EFINITE_BAG \\
 & A_27a) V1b2)))))) \\
 \end{aligned} \tag{14}$$

Assume the following.

$$(\forall V0n \in ty_2Enum_2Enum.((ap (ap c_2Ebag_2EBAG_GEN_PROD (c_2Ebag_2EMPTY_BAG ty_2Enum_2Enum)) V0n) = V0n)) \quad (15)$$

Assume the following.

$$\begin{aligned} & (\forall V0b \in (ty_2Enum_2Enum^{ty_2Enum_2Enum}).((p (ap (c_2Ebag_2EFINITE_BAG ty_2Enum_2Enum) V0b)) \Rightarrow (\forall V1x \in ty_2Enum_2Enum.(\forall V2a \in ty_2Enum_2Enum.((ap (ap c_2Ebag_2EBAG_GEN_PROD (ap (ap (c_2Ebag_2EBAG_INSERT ty_2Enum_2Enum) V1x) V0b)) V2a) = (ap (ap c_2Earithmetric_2E_2A V1x) (ap (ap c_2Ebag_2EBAG_GEN_PROD V0b) V2a))))))) \\ & \quad (16) \end{aligned}$$

Assume the following.

$$True \quad (17)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & \quad True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((\\ & \quad (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t))))))) \quad (18) \end{aligned}$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t)) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & \quad ((\neg False) \Leftrightarrow True)))) \quad (19)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0x \in A_27a.((V0x = V0x) \Leftrightarrow \\ & \quad True)) \quad (20)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0x \in A_27a.(\forall V1y \in \\ & \quad A_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (21)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & \quad (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg \\ & \quad (p V0t))))))) \quad (22) \end{aligned}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V0A) \vee (\\ & \quad (p V1B) \wedge (p V2C))) \Leftrightarrow (((p V0A) \vee (p V1B)) \wedge ((p V0A) \vee (p V2C))))))) \quad (23)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow \\ & \quad ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in 2. (\forall V1x_27 \in 2. (\forall V2y \in 2. (\forall V3y_27 \in \\ & 2. (((p V0x) \Leftrightarrow (p V1x_27)) \wedge ((p V1x_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_27)))))) \Rightarrow \end{aligned} \quad (25)$$

$$((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_27) \Rightarrow ((p V3y_27)))))))$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (26)$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (27)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg(p V0A) \vee (p V1B)) \Rightarrow False) \Leftrightarrow \\ & ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg(\neg(p V0A)) \vee (p V1B)) \Rightarrow False) \Leftrightarrow \\ & ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \end{aligned} \quad (29)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (30)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\ & (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg(p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\ & ((\neg(p V1q)) \vee (\neg(p V0p))))))))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\ & (p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee \\ & ((\neg(p V0p)) \wedge ((p V2r) \vee (\neg(p V0p))))))))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\ & (p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \wedge ((p V0p) \vee (\neg(p V2r)))) \wedge \\ & ((p V1q) \vee ((p V2r) \vee (\neg(p V0p))))))))))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\ & (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p))))))) \end{aligned} \quad (34)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p)))) \quad (35)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q))))) \quad (36)$$

Theorem 1

$$\begin{aligned} & (\forall V0b1 \in (ty_2Enum_2Enum^{ty_2Enum_2Enum}). ((p (ap (c_2Ebag_2EFINITE_BAG \\ & ty_2Enum_2Enum) V0b1)) \Rightarrow (\forall V1b2 \in (ty_2Enum_2Enum^{ty_2Enum_2Enum}). \\ & (\forall V2a \in ty_2Enum_2Enum. (\forall V3b \in ty_2Enum_2Enum. (\\ & (p (ap (c_2Ebag_2EFINITE_BAG ty_2Enum_2Enum) V1b2)) \Rightarrow ((ap (ap \\ & c_2Ebag_2EBAG_GEN_PROD (ap (ap (c_2Ebag_2EBAG_UNION ty_2Enum_2Enum) \\ & V0b1) V1b2)) (ap (ap c_2Earithmetic_2E_2A V2a) V3b)) = (ap (ap c_2Earithmetic_2E_2A \\ & (ap (ap c_2Ebag_2EBAG_GEN_PROD V0b1) V2a)) (ap (ap c_2Ebag_2EBAG_GEN_PROD \\ & V1b2) V3b)))))))))) \end{aligned}$$