thm_2Ebag_2EBAG__IMAGE__FINITE__UNION (TMQpvGTEtEwFN3DNiqmzRrsBgf5kXFSwd9n)

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Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let $c\_2Earithmetic\_2E\_2B : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{2}$$

Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V$

Definition 3 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 4 *We define* c_2Ebag_2EBAG__UNION *to be* $\lambda A\_27a : \iota.\lambda V0b \in (ty\_2Enum\_2Enum^{A\_27a}).\lambda V1c$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{3}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{4}$$

Definition 5 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 6 *We define* c_2Ecombin_2EK *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x))$

Definition 7 *We define* c_2Ebag_2EEMPTY__BAG *to be* $\lambda A\_27a : \iota.(ap\ (\mathsf{c\_2Ecombin\_2EK}\ ty\_2Enum\_2Enu$

Definition 8 *We define* c_2Earithmetic_2EZERO *to be* c_2Enum_2E0.

Let $c\_2Enum\_2EREP\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EREP\_\_num \in (omega^{ty\_2Enum\_2Enum}) \quad (5)$$

Let $c\_2Enum\_2ESUC\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2ESUC\_\_REP \in (omega^{omega}) \quad (6)$$

Definition 9 *We define* c_2Enum_2ESUC *to be* $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_\_num\ ($

Definition 10 *We define* c_2Earithmetic_2EBIT1 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Definition 11 *We define* c_2Earithmetic_2ENUMERAL *to be* $\lambda V0x \in ty\_2Enum\_2Enum.V0x$.

Definition 12 *We define* c_2Ebool_2EF *to be* $(ap\ (\text{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 13 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 14 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\text{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in$

Definition 15 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^{A}.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 16 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Definition 17 *We define* c_2Ebag_2EBAG__INSERT *to be* $\lambda A\_27a : \iota.\lambda V0e \in A\_27a.\lambda V1b \in (ty\_2Enum\_2E$

Definition 18 *We define* c_2Ebag_2EBAG__FILTER *to be* $\lambda A\_27a : \iota.\lambda V0P \in (2^{A\_27a}).\lambda V1b \in (ty\_2Enum\_$

Let $c\_2Ebag\_2EBAG\_\_CARD : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ebag\_2EBAG\_\_CARD\ A\_27a \in (ty\_2Enum\_2Enum^{(ty\_2Enum\_2Enum^{A\_27a})} \quad (7)$$

Definition 19 *We define* c_2Ebag_2EFINITE__BAG *to be* $\lambda A\_27a : \iota.\lambda V0b \in (ty\_2Enum\_2Enum^{A\_27a}).(ap$

Definition 20 *We define* c_2Ebool_2ELET *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0f \in (A\_27b^{A\_27a}).(\lambda V1x \in A\_27$

Definition 21 *We define* c_2Ebag_2EBAG__IMAGE *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27a^{A\_27b}).\lambda V1b$

Definition 22 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \text{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \text{c\_2Ebool\_2E}$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0e \in A\_27a.(\forall V1b1 \in (ty\_2Enum\_2Enum^{A\_27a}).(\forall V2b2 \in (ty\_2Enum\_2Enum^{A\_27a}).(((ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27a)\ (ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_INSERT}\ A\_27a)\ V0e)\ V1b1))\ V2b2) = (ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_INSERT}\ A\_27a)\ V0e)\ (ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27a)\ V1b1)\ V2b2))) \wedge ((ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27a)\ V1b1)\ (ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_INSERT}\ A\_27a)\ V0e)\ V2b2)) = (ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_INSERT}\ A\_27a)\ V0e)\ (ap\ (ap\ (\text{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27a)\ V1b1)\ V2b2)))))) \quad (8)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c.\ nonempty\ A\_27c \Rightarrow ((\forall V0b \in (ty\_2Enum\_2Enum^{A\_27a}).((ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27a)\ V0b)\ (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27a)) = V0b)) \wedge ((\forall V1b \in (ty\_2Enum\_2Enum^{A\_27b}).((ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27b)\ (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27b))\ V1b) = V1b)) \wedge (\forall V2b1 \in (ty\_2Enum\_2Enum^{A\_27c}).(\forall V3b2 \in (ty\_2Enum\_2Enum^{A\_27c}).(((ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27c)\ V2b1)\ V3b2) = (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27c)) \Leftrightarrow ((V2b1 = (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27c)) \wedge (V3b2 = (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27c))))))))) \tag{9}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{(ty\_2Enum\_2Enum^{A\_27a})}).(((p\ (ap\ V0P\ (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27a))) \wedge (\forall V1b \in (ty\_2Enum\_2Enum^{A\_27a}).(((p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ V1b)) \wedge (p\ (ap\ V0P\ V1b))) \Rightarrow (\forall V2e \in A\_27a.(p\ (ap\ V0P\ (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_INSERT}\ A\_27a)\ V2e)\ V1b))))))) \Rightarrow (\forall V3b \in (ty\_2Enum\_2Enum^{A\_27a}).((p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ V3b)) \Rightarrow (p\ (ap\ V0P\ V3b)))))) \tag{10}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0b1 \in (ty\_2Enum\_2Enum^{A\_27a}).(\forall V1b2 \in (ty\_2Enum\_2Enum^{A\_27a}).((p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27a)\ V0b1)\ V1b2))) \Leftrightarrow ((p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ V0b1)) \wedge (p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ V1b2)))))) \tag{11}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0f \in (A\_27b^{A\_27a}).((ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_IMAGE}\ A\_27b\ A\_27a)\ V0f)\ (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27a)) = (\mathsf{c\_2Ebag\_2EEMPTY\_\_BAG}\ A\_27b))) \tag{12}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0b \in (ty\_2Enum\_2Enum^{A\_27a}).(\forall V1f \in (A\_27b^{A\_27a}).(\forall V2e \in A\_27a.((p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ V0b)) \Rightarrow ((ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_IMAGE}\ A\_27b\ A\_27a)\ V1f)\ (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_INSERT}\ A\_27a)\ V2e)\ V0b)) = (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_INSERT}\ A\_27b)\ (ap\ V1f\ V2e))\ (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_IMAGE}\ A\_27b\ A\_27a)\ V1f)\ V0b))))))) \tag{13}$$

Assume the following.

$$True \tag{14}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \tag{15}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge ((((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t))))))) \tag{16}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))))) \tag{17}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{18}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{19}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg (p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (p\ V0t))))))) \tag{20}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p\ V0t1) \Rightarrow ((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \tag{21}$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x\_27 \in 2.(\forall V2y \in 2.(\forall V3y\_27 \in 2.((((p\ V0x) \Leftrightarrow (p\ V1x\_27)) \wedge ((p\ V1x\_27) \Rightarrow ((p\ V2y) \Leftrightarrow (p\ V3y\_27)))) \Rightarrow (((p\ V0x) \Rightarrow (p\ V2y)) \Leftrightarrow ((p\ V1x\_27) \Rightarrow (p\ V3y\_27)))))))) \tag{22}$$

Theorem 1

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0b1 \in (ty\_2Enum\_2Enum^{A\_27a}).(\forall V1b2 \in (ty\_2Enum\_2Enum^{A\_27a}).(\forall V2f \in (A\_27b^{A\_27a}).(((p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ V0b1)) \wedge (p\ (ap\ (\mathsf{c\_2Ebag\_2EFINITE\_\_BAG}\ A\_27a)\ V1b2))) \Rightarrow ((ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_IMAGE}\ A\_27b\ A\_27a)\ V2f)\ (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27a)\ V0b1)\ V1b2)) = (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_UNION}\ A\_27b)\ (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_IMAGE}\ A\_27b\ A\_27a)\ V2f)\ V0b1))\ (ap\ (ap\ (\mathsf{c\_2Ebag\_2EBAG\_\_IMAGE}\ A\_27b\ A\_27a)\ V2f)\ V1b2)))))))$$