

thm\_2Ebag\_2ESUB\_\_BAG\_\_CARD (TMaWDbR-  
bjNZZneYw7znuD8gRoxLCvcfGFZq)

October 26, 2020

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (1)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (2)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (3)$$

**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Enum\_2E0$  to be ( $ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP$ ).

**Definition 3** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (4)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (5)$$

**Definition 4** We define  $c\_2Ebool\_2ET$  to be ( $ap\ (ap\ (c\_2Emin\_2E\_3D\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1x)$ )

**Definition 5** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}).(ap\ (ap\ (c\_2Emin\_2E\_3D\ (2^{A-27a}))\ (\lambda V1x \in 2.V1x)))$

**Definition 6** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. (ap\ c\_2Enum\_2EABS\_num\ m)$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (6)$$

**Definition 7** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap (ap c\_2Earithmetic\_2EBIT1 n) V0)$

**Definition 8** We define `c_2Earthmetic_2ENUMERAL` to be  $\lambda V0x \in ty\_2Enum\_2Enum. V0x$ .

**Definition 9** We define  $c\_2Ebool\_2EF$  to be  $(ap\ (c\_2Ebool\_2E_21\ 2)\ (\lambda V0t\in 2.V0t))$ .

**Definition 10** We define  $c_{\text{2Emin\_2E\_3D\_3D\_3E}}$  to be  $\lambda P \in 2.\lambda Q \in 2.\text{inj\_o} (p \ P \Rightarrow p \ Q)$  of type  $\iota$ .

**Definition 11** We define  $c_2$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2 V0) V1) t2))$

**Definition 12** We define  $c_{\text{Emin}}.40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p \text{ (ap } P \text{ } x)) \text{ then } (\lambda x.x \in A \wedge_P$  of type  $\iota \Rightarrow \iota$ .

**Definition 13** We define  $c.2Ebool\_2ECOND$  to be  $\lambda A.27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A.27a.(\lambda V2t2 \in A.27a.($

**Definition 14** We define  $c.2EBag\_2EBAG\_INSERT$  to be  $\lambda A.27a : \iota.\lambda V0e \in A.27a.\lambda V1b \in (ty\_2Enum\_2E$

**Definition 15** We define  $c_2Ecombin\_2EK$  to be  $\lambda A.\_27a : \iota.\lambda A.\_27b : \iota.(\lambda V0x \in A.\_27a.(\lambda V1y \in A.\_27b.V0x))$

**Definition 16** We define  $c\_2Ebag\_2EEMPTY\_BAG$  to be  $\lambda A\_27a : \iota. (ap\ (c\_2Ecombin\_2EK\ ty\_2Enum\_2En$

**Definition 17** We define  $c_2EBag\_EFINITE\_BAG$  to be  $\lambda A.\lambda 27a : \iota.\lambda V0b \in (ty\_2Enum\_2Enum^A)^{27a}.$  (ap

Let  $c\_2EBag\_2EBAG\_CARD : \iota \Rightarrow \iota$  be given. Assume the following.

$\forall A \in \mathcal{A}, \exists a \text{ nonempty } A \ni a \in E \backslash \{a\} \text{ s.t. } a \in B \cap C$ .  $\text{CARD}(A) \geq 2$

(7)

Let  $c_2Earithmetic_2E_2D : \iota$  be given. Assume the following.

$$\frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \right) \left( \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \right) = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \right)^2$$

© 2024 All rights reserved. This material may not be published, reproduced, broadcast, rewritten, or redistributed without permission from the author.

**Definition 19** We define  $\mathcal{C}_{\text{ZEBAG-ZEBAG-SECTION}}$  to be  $\lambda A_1 \lambda A_2 \lambda x. \lambda y. \text{let } z = (\text{eg\_ZEBAGname\_ZEBAGname}) \text{ in } x \cdot y \cdot z$

**Definition 16** We define  $\text{CLOSURE}_P$  to be  $(\forall x \in \Sigma)(x_P \in \text{CLOSURE}_P \iff x \in \text{CLOSURE})$ .

**Definition 24** We define  $\text{C}_2\text{EBCS}(2,2,3)$  to be  $X \in \mathbb{R}^{2 \times 2 \times 3}$  s.t.  $(X, 0) \in \text{C}_2\text{EBCS}(2,2,3)$  and  $(X, 0) \vdash_{\text{C}_2\text{EBCS}(2,2,3)} (ap \vee b) \rightarrow (ap \wedge \text{EMIN}(2,2,3))$ .

**Definition 22** We define  $\text{C}_2\text{Ephm}[\text{rec}_2\text{Ephm}]$  to be  $\lambda v \; sm \in ig\_2\text{Ephm}[\text{Ephm}].\lambda v \; tn \in ig\_2\text{Ephm}[\text{Ephm}]$

**Definition 23** We define  $c_2$ Earthm $_2$ SL to be  $\lambda v\; sm \in cg\_2Earthm\_2Earthm.\lambda v\; tn \in cg\_2Earthm\_2Earthm.$

**Definition 24** We define  $C_{\leq LBSU(\leq L-SC, 2)}$  to be  $(\forall v \in T \in \mathbb{Z}. (\forall v' \in T \in \mathbb{Z}. (ap(C_{\leq LBSU(\leq L-SC, 2)})) \wedge (\forall v'' \in T \in \mathbb{Z}.$

**Definition 23** We define  $c_{\text{ZLArithmetc}} \in \text{SL-SD}$  to be  $\lambda x \ ym \in ig_{\text{ZLArithmetc}} \lambda x \ ym \in ig_{\text{ZLArithmetc}}$

**Definition 26** We define  $C_{\text{ZEBAG-ZEBAG-INV}}$  to be  $\lambda A_{\text{ZEBAG}} : \lambda V \. B \in A_{\text{ZEBAG}}. \lambda V \. \text{Inv} \in \text{ig-ZEBAG-INV}$

**Definition 27** We define  $c\_2Ebag\_2ESUB\_BAG$  to be  $\lambda A\_27a : \iota. \lambda V0b1 \in (ty\_2Enum\_2Enum^A)^{27a}. \lambda V1b1 \in (ty\_2Enum\_2Enum^{V0b1})^{27a}$ .

Let  $c\_2Earithmetic\_2EEVEN : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2EEVEN \in (2^{ty\_2Enum\_2Enum}) \quad (9)$$

Let  $c\_2Earithmetic\_2EODD : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2EODD \in (2^{ty\_2Enum\_2Enum}) \quad (10)$$

**Definition 28** We define  $c\_2Eprim\_rec\_2EPRE$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. (ap (ap (ap (ap (c\_2Ebool\_2Bool)))) V0m))$ .

Let  $c\_2Earithmetic\_2EEEXP : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2EEEXP \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (11)$$

Let  $c\_2Earithmetic\_2E\_2A : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2A \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (12)$$

**Definition 29** We define  $c\_2Enumeral\_2EiiSUC$  to be  $\lambda V0n \in ty\_2Enum\_2Enum. (ap c\_2Enum\_2ESUC (ap (ap (ap (ap (c\_2Ebool\_2Bool)))) V0n)))$ .

**Definition 30** We define  $c\_2Enumeral\_2EiZ$  to be  $\lambda V0x \in ty\_2Enum\_2Enum. V0x$ .

**Definition 31** We define  $c\_2Earithmetic\_2EBIT2$  to be  $\lambda V0n \in ty\_2Enum\_2Enum. (ap (ap c\_2Earithmetic\_2EBIT2 V0n))$ .

**Definition 32** We define  $c\_2Earithmetic\_2E\_3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. \lambda V1n \in ty\_2Enum\_2Enum. (ap (ap c\_2Eprim\_rec\_2EPRE (ap (ap (ap (ap (c\_2Ebool\_2Bool)))) V0m))) V1n)$ .

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & ((ap (ap c\_2Earithmetic\_2E\_2B c\_2Enum\_2E0) V0m) = V0m) \wedge (((ap ( \\ & ap (ap c\_2Earithmetic\_2E\_2B V0m) c\_2Enum\_2E0) = V0m) \wedge (((ap (ap c\_2Earithmetic\_2E\_2B \\ & (ap c\_2Enum\_2ESUC V0m)) V1n) = (ap c\_2Enum\_2ESUC (ap (ap c\_2Earithmetic\_2E\_2B \\ & V0m) V1n))) \wedge ((ap (ap c\_2Earithmetic\_2E\_2B V0m) (ap c\_2Enum\_2ESUC \\ & V1n)) = (ap c\_2Enum\_2ESUC (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n))))))) \\ & (13) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n) = (ap (ap c\_2Earithmetic\_2E\_2B \\ & V1n) V0m)))) \\ & (14) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & \forall V2p \in ty\_2Enum\_2Enum. ((ap (ap c\_2Earithmetic\_2E\_2B V0m) \\ & (ap (ap c\_2Earithmetic\_2E\_2B V1n) V2p)) = (ap (ap c\_2Earithmetic\_2E\_2B \\ & (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n)) V2p))))))) \\ & (15) \end{aligned}$$

Assume the following.

$$(\forall V0n \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Earithmetic\_2E\_3C\_3D c\_2Enum\_2E0) V0n))) \quad (16)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2A c\_2Enum\_2E0) V0m) = c\_2Enum\_2E0) \wedge \\ & (((ap (ap c\_2Earithmetic\_2E\_2A V0m) c\_2Enum\_2E0) = c\_2Enum\_2E0) \wedge \\ & (((ap (ap c\_2Earithmetic\_2E\_2A (ap c\_2Earithmetic\_2ENUMERAL \\ & (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))) V0m) = V0m) \wedge \\ & (((ap (ap c\_2Earithmetic\_2E\_2A V0m) (ap c\_2Earithmetic\_2ENUMERAL \\ & (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))) = V0m) \wedge \\ & ((ap (ap c\_2Earithmetic\_2E\_2A (ap c\_2Enum\_2ESUC V0m)) V1n) = (ap \\ & (ap c\_2Earithmetic\_2E\_2B (ap (ap c\_2Earithmetic\_2E\_2A V0m) V1n)) \\ & V1n) \wedge ((ap (ap c\_2Earithmetic\_2E\_2A V0m) (ap c\_2Enum\_2ESUC V1n)) = \\ & (ap (ap c\_2Earithmetic\_2E\_2B V0m) (ap (ap c\_2Earithmetic\_2E\_2A \\ & V0m) V1n))))))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.(\forall V2p \in ty\_2Enum\_2Enum.(((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\ & V0m) V1n)) \wedge (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V1n) V2p))) \Rightarrow (p ( \\ & ap (ap c\_2Earithmetic\_2E\_3C\_3D V0m) V2p))))))) \end{aligned} \quad (18)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V0m) V0m))) \quad (19)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.(\forall V2p \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\ & (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n)) (ap (ap c\_2Earithmetic\_2E\_2B \\ & V0m) V2p))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V1n) V2p))))))) \end{aligned} \quad (20)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.(\neg(p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V0m) V1n)) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\ & (ap c\_2Enum\_2ESUC V1n)) V0m)))))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} & (\forall V0n \in ty\_2Enum\_2Enum.((ap c\_2Enum\_2ESUC V0n) = (ap (ap \\ & c\_2Earithmetic\_2E\_2B (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 \\ & c\_2Earithmetic\_2EZERO))) V0n))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. nonempty A_{27a} \Rightarrow (\forall V0e \in A_{27a}. (\forall V1b1 \in \\
& (ty\_2Enum\_2Enum^{A_{27a}}). (\forall V2b2 \in (ty\_2Enum\_2Enum^{A_{27a}}). \\
& ((ap (ap (c\_2Ebag\_2EBAG\_UNION A_{27a}) (ap (ap (c\_2Ebag\_2EBAG\_INSERT \\
& A_{27a}) V0e) V1b1)) V2b2) = (ap (ap (c\_2Ebag\_2EBAG\_INSERT A_{27a}) \\
& V0e) (ap (ap (c\_2Ebag\_2EBAG\_UNION A_{27a}) V1b1) V2b2))) \wedge ((ap ( \\
& ap (c\_2Ebag\_2EBAG\_UNION A_{27a}) V1b1) (ap (ap (c\_2Ebag\_2EBAG\_INSERT \\
& A_{27a}) V0e) V2b2)) = (ap (ap (c\_2Ebag\_2EBAG\_INSERT A_{27a}) V0e) \\
& (ap (ap (c\_2Ebag\_2EBAG\_UNION A_{27a}) V1b1) V2b2))))))) \\
& (23)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. nonempty A_{27a} \Rightarrow \forall A_{27b}. nonempty A_{27b} \Rightarrow \forall A_{27c}. \\
& nonempty A_{27c} \Rightarrow ((\forall V0b \in (ty\_2Enum\_2Enum^{A_{27a}}). ((ap (ap \\
& (c\_2Ebag\_2EBAG\_UNION A_{27a}) V0b) (c\_2Ebag\_2EEMPTY\_BAG A_{27a})) = \\
& V0b)) \wedge ((\forall V1b \in (ty\_2Enum\_2Enum^{A_{27b}}). ((ap (ap (c\_2Ebag\_2EBAG\_UNION \\
& A_{27b}) (c\_2Ebag\_2EEMPTY\_BAG A_{27b})) V1b) = V1b)) \wedge (\forall V2b1 \in \\
& (ty\_2Enum\_2Enum^{A_{27c}}). (\forall V3b2 \in (ty\_2Enum\_2Enum^{A_{27c}}). \\
& ((ap (ap (c\_2Ebag\_2EBAG\_UNION A_{27c}) V2b1) V3b2) = (c\_2Ebag\_2EEMPTY\_BAG \\
& A_{27c})) \Leftrightarrow ((V2b1 = (c\_2Ebag\_2EEMPTY\_BAG A_{27c})) \wedge (V3b2 = (c\_2Ebag\_2EEMPTY\_BAG \\
& A_{27c}))))))) \\
& (24)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. nonempty A_{27a} \Rightarrow (\forall V0P \in (2^{(ty\_2Enum\_2Enum^{A_{27a}})}). \\
& (((p (ap V0P (c\_2Ebag\_2EEMPTY\_BAG A_{27a}))) \wedge (\forall V1b \in (ty\_2Enum\_2Enum^{A_{27a}}). \\
& (((p (ap (c\_2Ebag\_2EFINITE\_BAG A_{27a}) V1b)) \wedge (p (ap V0P V1b))) \Rightarrow \\
& (\forall V2e \in A_{27a}. (p (ap V0P (ap (ap (c\_2Ebag\_2EBAG\_INSERT A_{27a}) \\
& V2e) V1b))))))) \Rightarrow (\forall V3b \in (ty\_2Enum\_2Enum^{A_{27a}}). ((p (ap \\
& (c\_2Ebag\_2EFINITE\_BAG A_{27a}) V3b)) \Rightarrow (p (ap V0P V3b))))))) \\
& (25)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. nonempty A_{27a} \Rightarrow (\forall V0b1 \in (ty\_2Enum\_2Enum^{A_{27a}}). \\
& (\forall V1b2 \in (ty\_2Enum\_2Enum^{A_{27a}}). ((p (ap (c\_2Ebag\_2EFINITE\_BAG \\
& A_{27a}) (ap (ap (c\_2Ebag\_2EBAG\_UNION A_{27a}) V0b1) V1b2))) \Leftrightarrow ((p \\
& (ap (c\_2Ebag\_2EFINITE\_BAG A_{27a}) V0b1)) \wedge (p (ap (c\_2Ebag\_2EFINITE\_BAG \\
& A_{27a}) V1b2))))))) \\
& (26)
\end{aligned}$$

Assume the following.

$$\begin{aligned} \forall A_27a. \text{nonempty } A_27a \Rightarrow & (((ap(c_2Ebag_2EBAG_CARD A_27a) \\ & (c_2Ebag_2EMPTY_BAG A_27a)) = c_2Enum_2E0) \wedge (\forall V0b \in ( \\ & ty_2Enum_2Enum^{A_27a}).((p(ap(c_2Ebag_2FINITE_BAG A_27a) \\ & V0b)) \Rightarrow (\forall V1e \in A_27a.((ap(c_2Ebag_2EBAG_CARD A_27a) \\ & ap(ap(c_2Ebag_2EBAG_INSERT A_27a) V1e) V0b)) = (ap(ap(c_2Earithmetic_2E_2B \\ & (ap(c_2Ebag_2EBAG_CARD A_27a) V0b)) (ap(c_2Earithmetic_2ENUMERAL \\ & (ap(c_2Earithmetic_2EBIT1 c_2Earithmetic_2EZERO))))))))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} \forall A_27a. \text{nonempty } A_27a \Rightarrow & (\forall V0b1 \in (ty_2Enum_2Enum^{A_27a}). \\ & (\forall V1b2 \in (ty_2Enum_2Enum^{A_27a}).((p(ap(ap(c_2Ebag_2ESUB_BAG \\ & A_27a) V0b1) V1b2)) \Rightarrow (V1b2 = (ap(ap(c_2Ebag_2EBAG_UNION A_27a) \\ & V0b1) (ap(ap(c_2Ebag_2EBAG_DIFF A_27a) V1b2) V0b1))))))) \end{aligned} \quad (28)$$

Assume the following.

$$True \quad (29)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p \\ & V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2))))) \quad (30)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (31)$$

Assume the following.

$$(\forall V0t \in 2.((p V0t) \vee (\neg(p V0t)))) \quad (32)$$

Assume the following.

$$\forall A_27a. \text{nonempty } A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in \\ & A_27a.(p V0t) \Leftrightarrow (p V0t))) \quad (33)$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \Rightarrow False) \Rightarrow (\neg(p V0t)))) \quad (34)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (35)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t))))))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge \\ & (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee \\ & (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t))))))) \end{aligned} \quad (38)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t)) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ ((\neg False) \Leftrightarrow True)))))) \quad (39)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow \\ True)) \quad (40)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in \\ A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (41)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg \\ & (p V0t))))))) \end{aligned} \quad (42)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\exists V1x \in \\ A\_27a.(p (ap V0P V1x)))) \Leftrightarrow (\forall V2x \in A\_27a.(\neg(p (ap V0P V2x))))))) \quad (43)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A\_27a}).((\forall V2x \in A\_27a.((p V0P) \vee (p (ap V1Q V2x)))) \Leftrightarrow ((p \\ & V0P) \vee (\forall V3x \in A\_27a.(p (ap V1Q V3x))))))) \end{aligned} \quad (44)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V0A) \vee ( \\ & (p V1B) \vee (p V2C))) \Leftrightarrow (((p V0A) \vee (p V1B)) \vee (p V2C))))))) \quad (45)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((p V0A) \vee (p V1B)) \Leftrightarrow ((p V1B) \vee \\ (p V0A)))))) \quad (46)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg(p V0A) \vee (\neg(p V1B)))) \wedge ((\neg((p V0A) \vee (p V1B))) \Leftrightarrow ((\neg(p V0A) \wedge (\neg(p V1B))))))) \quad (47)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((p V0A) \Rightarrow (p V1B)) \Leftrightarrow ((\neg(p V0A)) \vee (p V1B)))) \quad (48)$$

Assume the following.

$$(\forall V0t \in 2. (((p V0t) \Rightarrow False) \Leftrightarrow ((p V0t) \Leftrightarrow False))) \quad (49)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (50)$$

Assume the following.

$$\begin{aligned} &(\forall V0x \in 2. (\forall V1x\_27 \in 2. (\forall V2y \in 2. (\forall V3y\_27 \in \\ &2. (((((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))) \Rightarrow \\ &(((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27))))))) \end{aligned} \quad (51)$$

Assume the following.

$((\forall V0n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2B c\_2Enum\_2E0) V0n) = V0n)) \wedge ((\forall V1n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2B V1n) c\_2Enum\_2E0) = V1n)) \wedge ((\forall V2n \in ty\_2Enum\_2Enum.((\forall V3m \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2B ap c\_2Earithmetic\_2ENUMERAL V2n)) (ap c\_2Earithmetic\_2ENUMERAL V3m)) = (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Enumeral\_2EiZ (ap (ap c\_2Earithmetic\_2E\_2B V2n) V3m))))))) \wedge ((\forall V4n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2A c\_2Enum\_2E0) V4n) = c\_2Enum\_2E0)) \wedge ((\forall V5n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2A V5n) c\_2Enum\_2E0) = c\_2Enum\_2E0)) \wedge ((\forall V6n \in ty\_2Enum\_2Enum.((\forall V7m \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2A (ap c\_2Earithmetic\_2ENUMERAL V6n)) (ap c\_2Earithmetic\_2ENUMERAL V7m)) = (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2A V6n) V7m))))))) \wedge ((\forall V8n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2D c\_2Enum\_2E0) V8n) = c\_2Enum\_2E0)) \wedge ((\forall V9n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2D V9n) c\_2Enum\_2E0) = V9n)) \wedge ((\forall V10n \in ty\_2Enum\_2Enum.((\forall V11m \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2E\_2D (ap c\_2Earithmetic\_2ENUMERAL V10n)) (ap c\_2Earithmetic\_2ENUMERAL V11m)) = (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2E\_2D V10n) V11m))))))) \wedge ((\forall V12n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2EEEXP c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 V12n))) = c\_2Enum\_2E0)) \wedge ((\forall V13n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2EEEXP c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT2 V13n))) = c\_2Enum\_2E0)) \wedge ((\forall V14n \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2EEEXP V14n) c\_2Enum\_2E0) = (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO)))))) \wedge ((\forall V15n \in ty\_2Enum\_2Enum.((\forall V16m \in ty\_2Enum\_2Enum.((ap (ap c\_2Earithmetic\_2EEEXP (ap c\_2Earithmetic\_2ENUMERAL V15n)) (ap c\_2Earithmetic\_2ENUMERAL V16m)) = (ap c\_2Earithmetic\_2ENUMERAL (ap (ap c\_2Earithmetic\_2EEEXP V15n) V16m))))))) \wedge (((ap c\_2Enum\_2ESUC c\_2Enum\_2E0) = (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Earithmetic\_2EBIT1 c\_2Earithmetic\_2EZERO))) \wedge ((\forall V17n \in ty\_2Enum\_2Enum.((ap c\_2Enum\_2ESUC (ap c\_2Earithmetic\_2ENUMERAL V17n)) = (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Enum\_2ESUC V17n))))))) \wedge (((ap c\_2Eprim\_rec\_2EPRE c\_2Enum\_2E0) = c\_2Enum\_2E0) \wedge ((\forall V18n \in ty\_2Enum\_2Enum.((ap c\_2Eprim\_rec\_2EPRE (ap c\_2Earithmetic\_2ENUMERAL V18n)) = (ap c\_2Earithmetic\_2ENUMERAL (ap c\_2Eprim\_rec\_2EPRE V18n))))))) \wedge ((\forall V19n \in ty\_2Enum\_2Enum.(((ap c\_2Earithmetic\_2ENUMERAL V19n) = c\_2Enum\_2E0) \Leftrightarrow (V19n = c\_2Earithmetic\_2EZERO))) \wedge ((\forall V20n \in ty\_2Enum\_2Enum.((c\_2Enum\_2E0 = (ap c\_2Earithmetic\_2ENUMERAL V20n)) \Leftrightarrow (V20n = c\_2Earithmetic\_2EZERO))) \wedge ((\forall V21n \in ty\_2Enum\_2Enum.((\forall V22m \in ty\_2Enum\_2Enum.(((ap c\_2Earithmetic\_2ENUMERAL V21n) = (ap c\_2Earithmetic\_2ENUMERAL V22m)) \Leftrightarrow (V21n = V22m))))))) \wedge ((\forall V23n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL V23n)) \Leftrightarrow False))) \wedge ((\forall V24n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL V24n)) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Earithmetic\_2EZERO) V24n)))) \wedge ((\forall V25n \in ty\_2Enum\_2Enum.((\forall V26m \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL V25n)) (ap c\_2Earithmetic\_2ENUMERAL V26m)) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Enum\_2E0) (ap c\_2Earithmetic\_2ENUMERAL V25n) V26m))))))) \wedge ((\forall V27n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3E c\_2Enum\_2E0) V27n)) \Leftrightarrow False))) \wedge ((\forall V28n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3E (ap c\_2Earithmetic\_2ENUMERAL V28n)) c\_2Enum\_2E0) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Earithmetic\_2EZERO) V28n)))) \wedge ((\forall V29n \in ty\_2Enum\_2Enum.((\forall V30m \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3E c\_2Enum\_2E0) V29n)) (ap c\_2Earithmetic\_2ENUMERAL V30m)) \Leftrightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C c\_2Enum\_2E0) V29n))))))) \wedge ((\forall V31n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3D c\_2Enum\_2E0) V31n)) \Leftrightarrow True))) \wedge ((\forall V32n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3C c\_2Enum\_2E0) V32n)) \Leftrightarrow False))) \wedge ((p (ap (ap c\_2Earithmetic\_2E\_3D c\_2Enum\_2E0) V32n)) \Leftrightarrow True))) \wedge ((\forall V33n \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3D c\_2Enum\_2E0) V33n)) \Leftrightarrow False))) \wedge ((p (ap (ap c\_2Earithmetic\_2E\_3D c\_2Enum\_2E0) V33n)) \Leftrightarrow True)))$

Assume the following.

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. \\
& ((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D c\_2Earithmetic\_2EZERO) V0n)) \Leftrightarrow \\
& True) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT1 \\
& V0n)) c\_2Earithmetic\_2EZERO)) \Leftrightarrow False) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& (ap c\_2Earithmetic\_2EBIT2 V0n)) c\_2Earithmetic\_2EZERO)) \Leftrightarrow False) \wedge \\
& (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT1 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT1 V1m))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V0n) V1m))) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT1 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT2 V1m))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V0n) V1m))) \wedge (((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT2 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT1 V1m))) \Leftrightarrow (\neg(p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V1m) V0n)))) \wedge ((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap c\_2Earithmetic\_2EBIT2 \\
& V0n)) (ap c\_2Earithmetic\_2EBIT2 V1m))) \Leftrightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\
& V0n) V1m))))))))))) \\
\end{aligned} \tag{54}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \tag{55}$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \tag{56}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \\
\end{aligned} \tag{57}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \\
\end{aligned} \tag{58}$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \tag{59}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\
& (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg(p \\
& V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\
& ((\neg(p V1q)) \vee (\neg(p V0p))))))))))) \\
\end{aligned} \tag{60}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\
& (p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee \\
& (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p)))))))))) \\
\end{aligned} \tag{61}$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \\ & \end{aligned} \quad (62)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((\neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \\ & \end{aligned} \quad (63)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p))))))) \quad (64)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p)))) \quad (65)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q)))))) \quad (66)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V0p)))))) \quad (67)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))))) \quad (68)$$

Assume the following.

$$(\forall V0p \in 2. ((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \quad (69)$$

### Theorem 1

$$\begin{aligned} & \forall A\_27a. \text{nonempty } A\_27a \Rightarrow (\forall V0b1 \in (ty\_2Enum\_2Enum^A\_27a). \\ & (\forall V1b2 \in (ty\_2Enum\_2Enum^A\_27a). (((p (ap (c\_2Ebag\_2EFINITE\_BAG \\ A\_27a) V1b2)) \wedge (p (ap (ap (c\_2Ebag\_2ESUB\_BAG A\_27a) V0b1) V1b2))) \Rightarrow \\ & (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap (c\_2Ebag\_2EBAG\_CARD A\_27a) \\ V0b1)) (ap (c\_2Ebag\_2EBAG\_CARD A\_27a) V1b2))))))) \\ & \end{aligned}$$