

thm\_2Ecardinal\_2ECARD\_\_SUBSET\_\_EQ  
(TMF8HkzzJ7mSAiCZsEKmpDGzbYofMMxH5B2)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let  $c\_2Epred\_set\_2ECARD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epred\_set\_2ECARD\ A\_27a \in (ty\_2Enum\_2Enum^{(2^{A-27a})}) \tag{2}$$

**Definition 3** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A-27a}).(ap\ V1f\ V0x)))$

**Definition 4** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p\ P \Rightarrow p\ Q)$  of type  $\iota$ .

**Definition 5** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a}))$

**Definition 6** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21\ 2)) (\lambda V2t \in 2$

**Definition 7** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21\ 2)) (\lambda V2t \in 2$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{3}$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A-27b})^{A-27a}}) \tag{4}$$

**Definition 8** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC \\ A\_27a A\_27b \in ((2^{A\_27a})^{((ty\_2Epair\_2Eprod A\_27a 2)^{A\_27b})}) \quad (5)$$

**Definition 9** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap (c\_2Ebool\_2EF$

**Definition 10** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 11** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2EF)$ .

**Definition 12** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).(ap (c\_2Ebool\_2E\_21 2)$

**Definition 13** We define  $c\_2Ecardinal\_2EHAS\_SIZE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1n \in ty\_2Enum\_2E$

**Definition 14** We define  $c\_2Ebool\_2E\_27E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_2D\_2D\_2E V0t) c\_2Ebool\_2EF$

**Definition 15** We define  $c\_2Epred\_set\_2EDIFF$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2E$

**Definition 16** We define  $c\_2Epred\_set\_2ESUBSET$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap ($

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (6)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (7)$$

**Definition 17** We define  $c\_2Enum\_2E0$  to be  $(ap c\_2Enum\_2EABS\_num c\_2Enum\_2EZERO\_REP)$ .

**Definition 18** We define  $c\_2Epred\_set\_2EINTER$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2E$

**Definition 19** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2E$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (8)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.( \\ (ap (ap c\_2Earithmetic\_2E\_2B V0m) V1n) = (ap (ap c\_2Earithmetic\_2E\_2B \\ V1n) V0m)))) \quad (9)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & \quad \forall V2p \in ty\_2Enum\_2Enum. (((ap (ap c\_2Earithmic\_2E\_2B V0m) \\ & \quad V2p) = (ap (ap c\_2Earithmic\_2E\_2B V1n) V2p)) \Leftrightarrow (V0m = V1n)))))) \end{aligned} \quad (10)$$

Assume the following.

$$True \quad (11)$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p \\ & \quad V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \end{aligned} \quad (12)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (13)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in \\ & \quad A\_27a. (p V0t)) \Leftrightarrow (p V0t))) \end{aligned} \quad (14)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ & \quad (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ & \quad (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & \quad True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & \quad (p V0t) \Rightarrow False) \Leftrightarrow (\neg (p V0t)))))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2. ((\neg (\neg (p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & \quad ((\neg False) \Leftrightarrow True)))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow \\ & \quad True)) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in \\ & \quad A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \end{aligned} \quad (19)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (20)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg(p V0A)) \vee (\neg(p V1B)))) \wedge (((\neg(p V0A)) \vee (p V1B)) \Leftrightarrow ((\neg(p V0A)) \wedge (\neg(p V1B)))))) \quad (21)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (22)$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x_{.27} \in 2.(\forall V2y \in 2.(\forall V3y_{.27} \in 2.(((p V0x) \Leftrightarrow (p V1x_{.27})) \wedge ((p V1x_{.27}) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_{.27})))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_{.27}) \Rightarrow (p V3y_{.27})))))) \quad (23)$$

Assume the following.

$$\forall A_{.27a}.nonempty A_{.27a} \Rightarrow (\forall V0s \in (2^{A_{.27a}}).((p (ap (ap (c_{.2}Ecardinal_{.2}EHAS_{.2}SIZE A_{.27a}) V0s) c_{.2}Enum_{.2}E0)) \Leftrightarrow (V0s = (c_{.2}Epred_{.2}set_{.2}EEMPTY A_{.27a})))) \quad (24)$$

Assume the following.

$$\forall A_{.27a}.nonempty A_{.27a} \Rightarrow (\forall V0s \in (2^{A_{.27a}}).(\forall V1t \in (2^{A_{.27a}}).((V0s = V1t) \Leftrightarrow (\forall V2x \in A_{.27a}.((p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) V0s)) \Leftrightarrow (p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) V1t)))))) \quad (25)$$

Assume the following.

$$\forall A_{.27a}.nonempty A_{.27a} \Rightarrow (\forall V0x \in A_{.27a}.(\neg(p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V0x) (c_{.2}Epred_{.2}set_{.2}EEMPTY A_{.27a})))))) \quad (26)$$

Assume the following.

$$\forall A_{.27a}.nonempty A_{.27a} \Rightarrow (\forall V0s \in (2^{A_{.27a}}).(\forall V1t \in (2^{A_{.27a}}).(\forall V2x \in A_{.27a}.((p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) (ap (ap (c_{.2}Epred_{.2}set_{.2}EUNION A_{.27a}) V0s) V1t))) \Leftrightarrow ((p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) V0s)) \vee (p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) V1t)))))) \quad (27)$$

Assume the following.

$$\forall A_{.27a}.nonempty A_{.27a} \Rightarrow (\forall V0s \in (2^{A_{.27a}}).(\forall V1t \in (2^{A_{.27a}}).(\forall V2x \in A_{.27a}.((p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) (ap (ap (c_{.2}Epred_{.2}set_{.2}EINTER A_{.27a}) V0s) V1t))) \Leftrightarrow ((p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) V0s)) \wedge (p (ap (ap (c_{.2}Ebool_{.2}EIN A_{.27a}) V2x) V1t)))))) \quad (28)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ (2^{A\_27a}). (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a) \\ V2x)\ (ap\ (ap\ (c\_2Epred\_set\_2EDIFF\ A\_27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap\ ( \\ ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V0s)) \wedge (\neg(p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\ A\_27a)\ V2x)\ V1t)))))))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ (2^{A\_27a}). (((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V0s)) \wedge (p\ ( \\ ap\ (ap\ (c\_2Epred\_set\_2ESUBSET\ A\_27a)\ V1t)\ V0s))) \Rightarrow (p\ (ap\ (c\_2Epred\_set\_2EFINITE \\ A\_27a)\ V1t)))))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow ((ap\ (c\_2Epred\_set\_2ECARD\ A\_27a) \\ (c\_2Epred\_set\_2EEMPTY\ A\_27a)) = c\_2Enum\_2E0) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). ((p\ (ap \\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V0s)) \Rightarrow (\forall V1t \in (2^{A\_27a}). \\ ((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V1t)) \Rightarrow ((ap\ (ap\ c\_2Earithmetic\_2E\_2B \\ (ap\ (c\_2Epred\_set\_2ECARD\ A\_27a)\ (ap\ (ap\ (c\_2Epred\_set\_2EUNION \\ A\_27a)\ V0s)\ V1t)))\ (ap\ (c\_2Epred\_set\_2ECARD\ A\_27a)\ (ap\ (ap\ (c\_2Epred\_set\_2EINTER \\ A\_27a)\ V0s)\ V1t))) = (ap\ (ap\ c\_2Earithmetic\_2E\_2B\ (ap\ (c\_2Epred\_set\_2ECARD \\ A\_27a)\ V0s))\ (ap\ (c\_2Epred\_set\_2ECARD\ A\_27a)\ V1t)))))) \end{aligned} \quad (32)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (33)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (34)$$

Assume the following.

$$\begin{aligned} (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) \end{aligned} \quad (36)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (37)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \Leftrightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee ((p \ V1q) \vee (p \ V2r))) \wedge (((p \ V0p) \vee ((\neg( \\
& p \ V2r)) \vee (\neg(p \ V1q)))) \wedge (((p \ V1q) \vee ((\neg(p \ V2r)) \vee (\neg(p \ V0p)))) \wedge ((p \ V2r) \vee \\
& ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \wedge (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee ((\neg(p \ V1q)) \vee (\neg(p \ V2r)))) \wedge (((p \ V1q) \vee \\
& (\neg(p \ V0p))) \wedge ((p \ V2r) \vee (\neg(p \ V0p))))))
\end{aligned} \tag{39}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \vee (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (\neg(p \ V1q))) \wedge (((p \ V0p) \vee (\neg(p \ V2r))) \wedge \\
& ((p \ V1q) \vee ((p \ V2r) \vee (\neg(p \ V0p))))))
\end{aligned} \tag{40}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \Rightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge (((p \ V0p) \vee (\neg(p \ V2r))) \wedge (( \\
& \neg(p \ V1q)) \vee ((p \ V2r) \vee (\neg(p \ V0p))))))
\end{aligned} \tag{41}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p \ V0p) \Leftrightarrow (\neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee \\
& (p \ V1q)) \wedge ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))
\end{aligned} \tag{42}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow (p \ V0p))) \tag{43}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow (\neg(p \ V1q)))) \tag{44}$$

**Theorem 1**

$$\begin{aligned}
& \forall A\_27a. \text{nonempty } A\_27a \Rightarrow (\forall V0a \in (2^{A\_27a}). (\forall V1b \in \\
& (2^{A\_27a}). (((p \ (ap \ (c\_2Epred\_set\_2EFINITE \ A\_27a) \ V1b)) \wedge ((p \\
& (ap \ (ap \ (c\_2Epred\_set\_2ESUBSET \ A\_27a) \ V0a) \ V1b)) \wedge ((ap \ (c\_2Epred\_set\_2ECARD \\
& A\_27a) \ V0a) = (ap \ (c\_2Epred\_set\_2ECARD \ A\_27a) \ V1b)))) \Rightarrow (V0a = V1b)))
\end{aligned}$$