

thm_2Ecardinal_2EUNION__LE__ADD__C
(TMKdQAP-
kjkGspk7oYTTBFTLpw2dZyFXCF3i)

October 26, 2020

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2ET$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 4 We define $c_2Ebool_2E_2EF$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 6 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))$

Definition 7 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p$ of type $\iota \Rightarrow \iota$.

Definition 8 We define $c_2Ebool_2E_2COND$ to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.(ap$

Definition 9 We define $c_2Ebool_2E_2EIN$ to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap V1f V0x)))$

Definition 10 We define $c_2Epred_set_2EINJ$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (A_27b^{A_27a}).\lambda V1s \in (2^{A_27a})$

Definition 11 We define $c_2Ebool_2E_23F$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap V0P (ap (c_2Emin_2E_40$

Definition 12 We define $c_2Ecardinal_2Ecardleq$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0s1 \in (2^{A_27a}).\lambda V1s2 \in (2^{A_27b})$

Definition 13 We define $c_2Ebool_2E_27E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_27E$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Esum_2Esum A0 A1)$$

(1)

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum \\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \end{aligned} \quad (2)$$

Definition 14 We define c_2Esum_2EINR to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27b.(ap\ (c_2Esum_2EABS_sum\ e))$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod \\ A0\ A1) \end{aligned} \quad (3)$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod \\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \end{aligned} \quad (4)$$

Definition 15 We define c_2Epair_2E2C to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap\ (c_2Epair_2EABS_prod\ x\ y))$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epred_set_2EGSPEC \\ A_27a\ A_27b \in ((2^{A_27a})^{(ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b}}) \end{aligned} \quad (5)$$

Definition 16 We define c_2Esum_2EINL to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27a.(ap\ (c_2Esum_2EABS_sum\ e))$

Definition 17 We define $c_2Ebool_2E5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E21\ 2)\ t1\ t2)))$

Definition 18 We define $c_2Epred_set_2EUNION$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap\ (c_2Epair_2E2C\ s\ t))$

Definition 19 We define $c_2Ecardinal_2E2B_c$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27b}).(ap\ (c_2Epair_2E2C\ s\ t))$

Definition 20 We define $c_2Epred_set_2ESUBSET$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap\ (c_2Epair_2E2C\ s\ t))$

Definition 21 We define $c_2Ecombin_2Eo$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0f \in (A_27b^{A_27c}).\lambda V1g \in (A_27a^{A_27c}).(ap\ (c_2Epair_2E2C\ f\ g))$

Definition 22 We define $c_2Epred_set_2EIMAGE$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (A_27b^{A_27a}).\lambda V1s \in (A_27a^{A_27a}).(ap\ (c_2Epair_2E2C\ f\ s))$

Let $c_2Esum_2EOUTL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EOUTL \\ A_27a\ A_27b \in (A_27a^{(ty_2Esum_2Esum\ A_27a\ A_27b)}) \end{aligned} \quad (6)$$

Let $c_2Esum_2EOUTR : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EOUTR \\ A_27a\ A_27b \in (A_27b^{(ty_2Esum_2Esum\ A_27a\ A_27b)}) \end{aligned} \quad (7)$$

Assume the following.

$$True \quad (8)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. ((p V0t1) \Rightarrow (p V1t2)) \Rightarrow ((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))) \quad (9)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (10)$$

Assume the following.

$$(\forall V0t \in 2. ((p V0t) \vee (\neg(p V0t)))) \quad (11)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A_27a. (p V0t)) \Leftrightarrow (p V0t))) \quad (12)$$

Assume the following.

$$(\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))) \quad (13)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (14)$$

Assume the following.

$$((\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge ((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True)) \quad (15)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0x \in A_27a. (V0x = V0x)) \quad (16)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (17)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (18)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0t1 \in A.27a. (\forall V1t2 \in \\ A.27a. (((ap\ (ap\ (ap\ (c.2Ebool.2ECOND\ A.27a)\ c.2Ebool.2ET)\ V0t1) \\ V1t2) = V0t1) \wedge ((ap\ (ap\ (ap\ (c.2Ebool.2ECOND\ A.27a)\ c.2Ebool.2EF) \\ V0t1)\ V1t2) = V1t2)))))) \end{aligned} \quad (19)$$

Assume the following.

$$\forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). ((\neg(\exists V1x \in \\ A.27a. (p\ (ap\ V0P\ V1x)))))) \Leftrightarrow (\forall V2x \in A.27a. (\neg(p\ (ap\ V0P\ V2x)))))) \quad (20)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow (\\ \forall V0f \in (A.27b^{A.27a}). (\forall V1s \in (2^{A.27a}). (\forall V2t \in \\ (2^{A.27b}). ((p\ (ap\ (ap\ (c.2Epred_set.2ESUBSET\ A.27b)\ V2t)\ (ap \\ (ap\ (c.2Epred_set.2EIMAGE\ A.27a\ A.27b)\ V0f)\ V1s)))) \Rightarrow (p\ (ap\ (ap \\ (c.2Ecardinal.2Ecardleq\ A.27b\ A.27a)\ V2t)\ V1s)))))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A.27a}). (\forall V1t \in \\ (2^{A.27a}). (\forall V2x \in A.27a. ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a) \\ V2x)\ (ap\ (ap\ (c.2Epred_set.2EUNION\ A.27a)\ V0s)\ V1t)))) \Leftrightarrow ((p\ (ap \\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V2x)\ V0s)) \vee (p\ (ap\ (ap\ (c.2Ebool.2EIN \\ A.27a)\ V2x)\ V1t)))))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow (\\ \forall V0y \in A.27b. (\forall V1s \in (2^{A.27a}). (\forall V2f \in (A.27b^{A.27a}). \\ ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27b)\ V0y)\ (ap\ (ap\ (c.2Epred_set.2EIMAGE \\ A.27a\ A.27b)\ V2f)\ V1s)))) \Leftrightarrow (\exists V3x \in A.27a. ((V0y = (ap\ V2f\ V3x)) \wedge \\ (p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V3x)\ V1s)))))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow \forall A.27c. \\ nonempty\ A.27c \Rightarrow (\forall V0f \in (A.27c^{A.27b}). (\forall V1g \in (A.27b^{A.27a}). \\ (\forall V2s \in (2^{A.27a}). ((ap\ (ap\ (c.2Epred_set.2EIMAGE\ A.27a \\ A.27c)\ (ap\ (ap\ (c.2Ecombin.2Eo\ A.27a\ A.27c\ A.27b)\ V0f)\ V1g))\ V2s) = \\ (ap\ (ap\ (c.2Epred_set.2EIMAGE\ A.27b\ A.27c)\ V0f)\ (ap\ (ap\ (c.2Epred_set.2EIMAGE \\ A.27a\ A.27b)\ V1g)\ V2s)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \quad \forall V0f \in (A_27b^{A_27a}).(\forall V1s \in (2^{A_27a}).(\forall V2t \in \\ & (2^{A_27a}).((ap\ (ap\ (c_2Epred_set_2EIMAGE\ A_27a\ A_27b)\ V0f)\ (\\ & ap\ (ap\ (c_2Epred_set_2EUNION\ A_27a)\ V1s)\ V2t)) = (ap\ (ap\ (c_2Epred_set_2EUNION \\ & A_27b)\ (ap\ (ap\ (c_2Epred_set_2EIMAGE\ A_27a\ A_27b)\ V0f)\ V1s))\ (\\ & ap\ (ap\ (c_2Epred_set_2EIMAGE\ A_27a\ A_27b)\ V0f)\ V2t)))))) \\ & \hspace{15em} (25) \end{aligned}$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (26)$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (27)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \end{aligned} \quad (29)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (30)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ & (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg \\ & p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee \\ & ((\neg(p\ V1q)) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\ & (p\ V1q) \Rightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee (p\ V1q)) \wedge (((p\ V0p) \vee (\neg(p\ V2r))) \wedge ((\\ & \neg(p\ V1q)) \vee ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \end{aligned} \quad (32)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p) \Rightarrow (p\ V1q))) \Rightarrow (p\ V0p))) \quad (33)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p) \Rightarrow (p\ V1q))) \Rightarrow (\neg(p\ V1q)))) \quad (34)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V0p)))))) \quad (35)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))))) \quad (36)$$

Assume the following.

$$(\forall V0p \in 2.((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \quad (37)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \forall V0x \in A_27a.(\forall V1y \in A_27b.(\neg((ap\ (c_2Esum_2EINL \\ & A_27a\ A_27b)\ V0x) = (ap\ (c_2Esum_2EINR\ A_27a\ A_27b)\ V1y)))))) \end{aligned} \quad (38)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \forall V0x \in A_27a.((ap\ (c_2Esum_2EOUTL\ A_27a\ A_27b)\ (ap\ (c_2Esum_2EINL \\ & A_27a\ A_27b)\ V0x)) = V0x)) \end{aligned} \quad (39)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \forall V0x \in A_27b.((ap\ (c_2Esum_2EOUTR\ A_27a\ A_27b)\ (ap\ (c_2Esum_2EINR \\ & A_27a\ A_27b)\ V0x)) = V0x)) \end{aligned} \quad (40)$$

Theorem 1

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0s \in (2^{A_27a}).(\forall V1t \in \\ & (2^{A_27a}).(p\ (ap\ (ap\ (c_2Ecardinal_2Ecardleq\ A_27a\ (ty_2Esum_2Esum \\ & A_27a\ A_27a))\ (ap\ (ap\ (c_2Epred_set_2EUNION\ A_27a)\ V0s)\ V1t)) \\ & (ap\ (ap\ (c_2Ecardinal_2EB_c\ A_27a\ A_27a)\ V0s)\ V1t)))))) \end{aligned}$$