

thm\_2Ecardinal\_2Ecardeq\_\_bijns\_\_cong  
(TMdTEgpDqsdbzCgx7eyj8XYmzeuyv2rw4AL)

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**Definition 1** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.$ **if**  $(\exists x \in A.p (ap P x))$  **then** (the  $(\lambda x.x \in A \wedge p$   
of type  $\iota \Rightarrow \iota$ ).

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$   
of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A.\lambda a : \iota.(\lambda V0x \in A.\lambda a.(\lambda V1f \in (2^{A-27a}).(ap V1f V0x)))$

**Definition 4** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$   
of type  $\iota$ .

**Definition 5** We define  $c\_2Ebool\_2EET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 6** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a}))$

**Definition 7** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t)))$

**Definition 8** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A$

**Definition 9** We define  $c\_2Epred\_set\_2ESURJ$  to be  $\lambda A.\lambda a : \iota.\lambda A.\lambda b : \iota.\lambda V0f \in (A.\lambda b^{A-27a}).\lambda V1s \in (2^{A-27a})$

**Definition 10** We define  $c\_2Epred\_set\_2EINJ$  to be  $\lambda A.\lambda a : \iota.\lambda A.\lambda b : \iota.\lambda V0f \in (A.\lambda b^{A-27a}).\lambda V1s \in (2^{A-27a})$

**Definition 11** We define  $c\_2Epred\_set\_2EBIJ$  to be  $\lambda A.\lambda a : \iota.\lambda A.\lambda b : \iota.\lambda V0f \in (A.\lambda b^{A-27a}).\lambda V1s \in (2^{A-27a})$

**Definition 12** We define  $c\_2Ecardinal\_2Ecardeq$  to be  $\lambda A.\lambda a : \iota.\lambda A.\lambda b : \iota.\lambda V0s1 \in (2^{A-27a}).\lambda V1s2 \in (2^{A-27a})$

Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \tag{1}$$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \tag{2}$$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum \\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \end{aligned} \quad (3)$$

**Definition 13** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap\ (c\_2Esum\_2EABS\_sum\ e))$

Let  $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Eoption\_2Eoption\ A0) \quad (4)$$

Let  $c\_2Eoption\_2Eoption\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eoption\_2Eoption\_ABS\ A\_27a \in \\ ((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ ty\_2Eone\_2Eone)}) \end{aligned} \quad (5)$$

**Definition 14** We define  $c\_2Eoption\_2ESOME$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.(ap\ (c\_2Eoption\_2Eoption\_ABS\ x))$

Let  $c\_2Eoption\_2ETHE : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eoption\_2ETHE\ A\_27a \in (A\_27a^{(ty\_2Eoption\_2Eoption\ A\_27a)}) \quad (6)$$

**Definition 15** We define  $c\_2Ecombin\_2Eo$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in (A\_27b^{A\_27c}).\lambda V1g \in (A\_27c^{A\_27a}).\lambda V2h \in (A\_27a^{A\_27b}).\lambda V3i \in (A\_27b^{A\_27c}).\lambda V4j \in (A\_27c^{A\_27a}).\lambda V5k \in (A\_27a^{A\_27b}).\lambda V6l \in (A\_27b^{A\_27c}).\lambda V7m \in (A\_27c^{A\_27a}).\lambda V8n \in (A\_27a^{A\_27b}).\lambda V9o \in (A\_27b^{A\_27c}).\lambda V10p \in (A\_27c^{A\_27a}).\lambda V11q \in (A\_27a^{A\_27b}).\lambda V12r \in (A\_27b^{A\_27c}).\lambda V13s \in (A\_27c^{A\_27a}).\lambda V14t \in (A\_27a^{A\_27b}).\lambda V15u \in (A\_27b^{A\_27c}).\lambda V16v \in (A\_27c^{A\_27a}).\lambda V17w \in (A\_27a^{A\_27b}).\lambda V18x \in (A\_27b^{A\_27c}).\lambda V19y \in (A\_27c^{A\_27a}).\lambda V20z \in (A\_27a^{A\_27b}).\lambda V21aa \in (A\_27b^{A\_27c}).\lambda V22ab \in (A\_27c^{A\_27a}).\lambda V23ba \in (A\_27a^{A\_27b}).\lambda V24bb \in (A\_27b^{A\_27c}).\lambda V25ca \in (A\_27c^{A\_27a}).\lambda V26cb \in (A\_27a^{A\_27b}).\lambda V27cb \in (A\_27b^{A\_27c}).\lambda V28cc \in (A\_27c^{A\_27a}).\lambda V29ca \in (A\_27a^{A\_27b}).\lambda V30cb \in (A\_27b^{A\_27c}).\lambda V31cc \in (A\_27c^{A\_27a}).\lambda V32ca \in (A\_27a^{A\_27b}).\lambda V33cb \in (A\_27b^{A\_27c}).\lambda V34cc \in (A\_27c^{A\_27a}).\lambda V35ca \in (A\_27a^{A\_27b}).\lambda V36cb \in (A\_27b^{A\_27c}).\lambda V37cc \in (A\_27c^{A\_27a}).\lambda V38ca \in (A\_27a^{A\_27b}).\lambda V39cb \in (A\_27b^{A\_27c}).\lambda V40cc \in (A\_27c^{A\_27a}).\lambda V41ca \in (A\_27a^{A\_27b}).\lambda V42cb \in (A\_27b^{A\_27c}).\lambda V43cc \in (A\_27c^{A\_27a}).\lambda V44ca \in (A\_27a^{A\_27b}).\lambda V45cb \in (A\_27b^{A\_27c}).\lambda V46cc \in (A\_27c^{A\_27a}).\lambda V47ca \in (A\_27a^{A\_27b}).\lambda V48cb \in (A\_27b^{A\_27c}).\lambda V49cc \in (A\_27c^{A\_27a}).\lambda V50ca \in (A\_27a^{A\_27b}).\lambda V51cb \in (A\_27b^{A\_27c}).\lambda V52cc \in (A\_27c^{A\_27a}).\lambda V53ca \in (A\_27a^{A\_27b}).\lambda V54cb \in (A\_27b^{A\_27c}).\lambda V55cc \in (A\_27c^{A\_27a}).\lambda V56ca \in (A\_27a^{A\_27b}).\lambda V57cb \in (A\_27b^{A\_27c}).\lambda V58cc \in (A\_27c^{A\_27a}).\lambda V59ca \in (A\_27a^{A\_27b}).\lambda V60cb \in (A\_27b^{A\_27c}).\lambda V61cc \in (A\_27c^{A\_27a}).\lambda V62ca \in (A\_27a^{A\_27b}).\lambda V63cb \in (A\_27b^{A\_27c}).\lambda V64cc \in (A\_27c^{A\_27a}).\lambda V65ca \in (A\_27a^{A\_27b}).\lambda V66cb \in (A\_27b^{A\_27c}).\lambda V67cc \in (A\_27c^{A\_27a}).\lambda V68ca \in (A\_27a^{A\_27b}).\lambda V69cb \in (A\_27b^{A\_27c}).\lambda V70cc \in (A\_27c^{A\_27a}).\lambda V71ca \in (A\_27a^{A\_27b}).\lambda V72cb \in (A\_27b^{A\_27c}).\lambda V73cc \in (A\_27c^{A\_27a}).\lambda V74ca \in (A\_27a^{A\_27b}).\lambda V75cb \in (A\_27b^{A\_27c}).\lambda V76cc \in (A\_27c^{A\_27a}).\lambda V77ca \in (A\_27a^{A\_27b}).\lambda V78cb \in (A\_27b^{A\_27c}).\lambda V79cc \in (A\_27c^{A\_27a}).\lambda V80ca \in (A\_27a^{A\_27b}).\lambda V81cb \in (A\_27b^{A\_27c}).\lambda V82cc \in (A\_27c^{A\_27a}).\lambda V83ca \in (A\_27a^{A\_27b}).\lambda V84cb \in (A\_27b^{A\_27c}).\lambda V85cc \in (A\_27c^{A\_27a}).\lambda V86ca \in (A\_27a^{A\_27b}).\lambda V87cb \in (A\_27b^{A\_27c}).\lambda V88cc \in (A\_27c^{A\_27a}).\lambda V89ca \in (A\_27a^{A\_27b}).\lambda V90cb \in (A\_27b^{A\_27c}).\lambda V91cc \in (A\_27c^{A\_27a}).\lambda V92ca \in (A\_27a^{A\_27b}).\lambda V93cb \in (A\_27b^{A\_27c}).\lambda V94cc \in (A\_27c^{A\_27a}).\lambda V95ca \in (A\_27a^{A\_27b}).\lambda V96cb \in (A\_27b^{A\_27c}).\lambda V97cc \in (A\_27c^{A\_27a}).\lambda V98ca \in (A\_27a^{A\_27b}).\lambda V99cb \in (A\_27b^{A\_27c}).\lambda V100cc \in (A\_27c^{A\_27a}).$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod \\ A0\ A1) \end{aligned} \quad (7)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \end{aligned} \quad (8)$$

**Definition 16** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Epair\_2EABS\_prod\ x\ y))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC \\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}}) \end{aligned} \quad (9)$$

**Definition 17** We define  $c\_2Ecardinal\_2Ebijns$  to be  $\lambda A\_27a : \iota.\lambda V0A \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EGSPEC\ A\ A))$

**Definition 18** We define  $c\_2Eone\_2Eone$  to be  $(ap\ (c\_2Emin\_2E\_40\ ty\_2Eone\_2Eone))\ (\lambda V0x \in ty\_2Eone\_2Eone.2)$

**Definition 19** We define  $c\_2Ebool\_2EF$  to be  $(ap\ (c\_2Ebool\_2E\_21\ 2))\ (\lambda V0t \in 2.V0t)$ .

**Definition 20** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E$

**Definition 21** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap (c\_2Esum\_2EABS$

**Definition 22** We define  $c\_2Eoption\_2ENONE$  to be  $\lambda A\_27a : \iota.(ap (c\_2Eoption\_2Eoption\_ABS A\_27a) (c$

**Definition 23** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

**Definition 24** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in$

Assume the following.

$$True \tag{10}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \tag{11}$$

Assume the following.

$$(\forall V0t \in 2.((p V0t) \vee (\neg(p V0t)))) \tag{12}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p V0t)) \Leftrightarrow (p V0t))) \tag{13}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \tag{14}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \tag{15}$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True)))) \tag{16}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(V0x = V0x)) \tag{17}$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{18}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (19)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0f \in (A\_27b^{A\_27a}). (\forall V1g \in (A\_27b^{A\_27a}). ((V0f = V1g) \Leftrightarrow (\forall V2x \in A\_27a. ((ap\ V0f\ V2x) = (ap\ V1g\ V2x)))))) \quad (20)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg(p\ V0t)))))) \quad (21)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}). (\forall V1Q \in 2. (((\forall V2x \in A\_27a. (p\ (ap\ V0P\ V2x))) \wedge (p\ V1Q)) \Leftrightarrow (\forall V3x \in A\_27a. ((p\ (ap\ V0P\ V3x)) \wedge (p\ V1Q)))))) \quad (22)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0Q \in 2. (\forall V1P \in (2^{A\_27a}). ((\forall V2x \in A\_27a. ((p\ (ap\ V1P\ V2x)) \vee (p\ V0Q))) \Leftrightarrow ((\forall V3x \in A\_27a. (p\ (ap\ V1P\ V3x))) \vee (p\ V0Q)))))) \quad (23)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in 2. (\forall V1Q \in (2^{A\_27a}). ((\forall V2x \in A\_27a. ((p\ V0P) \vee (p\ (ap\ V1Q\ V2x)))) \Leftrightarrow ((p\ V0P) \vee (\forall V3x \in A\_27a. (p\ (ap\ V1Q\ V3x)))))) \quad (24)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p\ V0A) \vee (p\ V1B) \vee (p\ V2C)) \Leftrightarrow (((p\ V0A) \vee (p\ V1B)) \vee (p\ V2C)))))) \quad (25)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((p\ V0A) \vee (p\ V1B)) \Leftrightarrow ((p\ V1B) \vee (p\ V0A)))) \quad (26)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p\ V1B) \wedge (p\ V2C)) \vee (p\ V0A)) \Leftrightarrow (((p\ V1B) \vee (p\ V0A)) \wedge ((p\ V2C) \vee (p\ V0A)))))) \quad (27)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((p V0A) \Rightarrow (p V1B)) \Leftrightarrow ((\neg(p V0A)) \vee (p V1B)))))) \quad (28)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (29)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Leftrightarrow (p V1t2)) \Leftrightarrow (((p V0t1) \Rightarrow (p V1t2)) \wedge ((p V1t2) \Rightarrow (p V0t1)))))) \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty \ A\_27a \Rightarrow \forall A\_27b.nonempty \ A\_27b \Rightarrow ( \\ & \quad \forall V0b \in 2.(\forall V1f \in (A\_27b^{A\_27a}).(\forall V2g \in (A\_27b^{A\_27a}). \\ & \quad (\forall V3x \in A\_27a.((ap (ap (ap (ap (c\_2Ebool\_2ECOND (A\_27b^{A\_27a})) \\ & \quad V0b) V1f) V2g) V3x) = (ap (ap (ap (c\_2Ebool\_2ECOND A\_27b) V0b) (ap \\ & \quad V1f V3x)) (ap V2g V3x))))))))) \quad (31) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty \ A\_27a \Rightarrow \forall A\_27b.nonempty \ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in (A\_27b^{A\_27a}).(\forall V1b \in 2.(\forall V2x \in A\_27a. \\ & \quad (\forall V3y \in A\_27a.((ap V0f (ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) \\ & \quad V1b) V2x) V3y)) = (ap (ap (ap (c\_2Ebool\_2ECOND A\_27b) V1b) (ap V0f \\ & \quad V2x)) (ap V0f V3y))))))))) \quad (32) \end{aligned}$$

Assume the following.

$$2.(((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))) \Rightarrow 2.(((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27)))) \quad (33)$$

Assume the following.

$$2.(((p V2Q) \Rightarrow ((p V0P) \Leftrightarrow (p V1P\_27))) \wedge ((p V1P\_27) \Rightarrow ((p V2Q) \Leftrightarrow (p V3Q\_27)))) \Rightarrow 2.(((p V0P) \wedge (p V2Q)) \Leftrightarrow ((p V1P\_27) \wedge (p V3Q\_27)))) \quad (34)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in 2. \\ & \quad (\forall V2x \in A\_27a.(\forall V3x\_27 \in A\_27a.(\forall V4y \in A\_27a. \\ & \quad (\forall V5y\_27 \in A\_27a.(((p V0P) \Leftrightarrow (p V1Q)) \wedge (((p V1Q) \Rightarrow (V2x = V3x\_27)) \wedge \\ & \quad ((\neg(p V1Q)) \Rightarrow (V4y = V5y\_27)))))) \Rightarrow ((ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) \\ & \quad V0P) V2x) V4y) = (ap (ap (ap (c\_2Ebool\_2ECOND A\_27a) V1Q) V3x\_27 \\ & \quad V5y\_27))))))))) \quad (35) \end{aligned}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A-27a}). (\forall V1a \in A\_27a. ((\exists V2x \in A\_27a. ((V2x = V1a) \wedge (p\ (ap\ V0P\ V2x)))) \Leftrightarrow (p\ (ap\ V0P\ V1a)))))) \quad (36)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0f \in (2^{A-27a}). (\forall V1v \in A\_27a. ((\forall V2x \in A\_27a. ((V2x = V1v) \Rightarrow (p\ (ap\ V0f\ V2x)))) \Leftrightarrow (p\ (ap\ V0f\ V1v)))))) \quad (37)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((\forall V0t1 \in A\_27a. (\forall V1t2 \in A\_27a. ((ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ c\_2Ebool\_2ET)\ V0t1)\ V1t2) = V0t1))) \wedge (\forall V2t1 \in A\_27a. (\forall V3t2 \in A\_27a. ((ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ c\_2Ebool\_2EF)\ V2t1)\ V3t2) = V3t2)))))) \quad (38)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A-27a}). (\forall V1Q \in 2. (((\exists V2x \in A\_27a. (p\ (ap\ V0P\ V2x))) \Rightarrow (p\ V1Q)) \Leftrightarrow (\forall V3x \in A\_27a. ((p\ (ap\ V0P\ V3x)) \Rightarrow (p\ V1Q)))))) \wedge (((\exists V4x \in A\_27a. (p\ (ap\ V0P\ V4x)) \wedge (p\ V1Q)) \Leftrightarrow (\exists V5x \in A\_27a. ((p\ (ap\ V0P\ V5x)) \wedge (p\ V1Q)))))) \wedge (((p\ V1Q) \wedge (\exists V6x \in A\_27a. (p\ (ap\ V0P\ V6x)))) \Leftrightarrow (\exists V7x \in A\_27a. ((p\ V1Q) \wedge (p\ (ap\ V0P\ V7x)))))))))) \quad (39)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0s \in (2^{A-27a}). (\forall V1t \in (2^{A-27b}). ((p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardeq\ A\_27a\ A\_27b)\ V0s)\ V1t)) \Leftrightarrow (p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardeq\ A\_27b\ A\_27a)\ V1t)\ V0s)))))) \quad (40)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c.nonempty\ A\_27c \Rightarrow (\forall V0f \in (A\_27b^{A-27a}). (\forall V1g \in (A\_27a^{A-27c}). (\forall V2x \in A\_27c. ((ap\ (ap\ (ap\ (c\_2Ecombin\_2Eo\ A\_27c\ A\_27b\ A\_27a)\ V0f)\ V1g)\ V2x) = (ap\ V0f\ (ap\ V1g\ V2x)))))) \quad (41)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0opt \in (ty\_2Eoption\_2Eoption\ A\_27a). ((V0opt = (c\_2Eoption\_2ENONE\ A\_27a)) \vee (\exists V1x \in A\_27a. (V0opt = (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V1x)))))) \quad (42)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a. (\forall V1y \in \\ & A.27a. (((ap\ (c.2Eoption\_2ESOME\ A.27a)\ V0x) = (ap\ (c.2Eoption\_2ESOME \\ & A.27a)\ V1y)) \Leftrightarrow (V0x = V1y)))) \end{aligned} \quad (43)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a. ((ap\ (c.2Eoption\_2ETHE \\ & A.27a)\ (ap\ (c.2Eoption\_2ESOME\ A.27a)\ V0x)) = V0x)) \end{aligned} \quad (44)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0P \in 2. (\forall V1x \in A.27a. \\ & (\forall V2y \in A.27a. (((ap\ (ap\ (ap\ (c.2Ebool\_2ECOND\ (ty.2Eoption\_2Eoption \\ & A.27a))\ V0P)\ (ap\ (c.2Eoption\_2ESOME\ A.27a)\ V1x))\ (c.2Eoption\_2ENONE \\ & A.27a)) = (c.2Eoption\_2ENONE\ A.27a)) \Leftrightarrow (\neg(p\ V0P))) \wedge (((ap\ (ap\ ( \\ & ap\ (c.2Ebool\_2ECOND\ (ty.2Eoption\_2Eoption\ A.27a))\ V0P)\ (c.2Eoption\_2ENONE \\ & A.27a))\ (ap\ (c.2Eoption\_2ESOME\ A.27a)\ V1x)) = (c.2Eoption\_2ENONE \\ & A.27a)) \Leftrightarrow (p\ V0P))) \wedge (((ap\ (ap\ (ap\ (c.2Ebool\_2ECOND\ (ty.2Eoption\_2Eoption \\ & A.27a))\ V0P)\ (ap\ (c.2Eoption\_2ESOME\ A.27a)\ V1x))\ (c.2Eoption\_2ENONE \\ & A.27a)) = (ap\ (c.2Eoption\_2ESOME\ A.27a)\ V2y)) \Leftrightarrow ((p\ V0P) \wedge (V1x = V2y))) \wedge \\ & (((ap\ (ap\ (ap\ (c.2Ebool\_2ECOND\ (ty.2Eoption\_2Eoption\ A.27a)) \\ & V0P)\ (c.2Eoption\_2ENONE\ A.27a))\ (ap\ (c.2Eoption\_2ESOME\ A.27a) \\ & V1x)) = (ap\ (c.2Eoption\_2ESOME\ A.27a)\ V2y)) \Leftrightarrow ((\neg(p\ V0P)) \wedge (V1x = \\ & V2y))))))))) \end{aligned} \quad (45)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\ & \forall V0x \in A.27a. (\forall V1y \in A.27b. (\forall V2a \in A.27a. (\forall V3b \in \\ & A.27b. (((ap\ (ap\ (c.2Epair\_2E\_2C\ A.27a\ A.27b)\ V0x)\ V1y) = (ap\ (ap \\ & (c.2Epair\_2E\_2C\ A.27a\ A.27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b)))))) \end{aligned} \quad (46)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\ & \forall V0f \in ((ty.2Epair\_2Eprod\ A.27a\ 2)^{A.27b}). (\forall V1v \in \\ & A.27a. ((p\ (ap\ (ap\ (c.2Ebool\_2EIN\ A.27a)\ V1v)\ (ap\ (c.2Epred\_set.2EGSPEC \\ & A.27a\ A.27b)\ V0f))) \Leftrightarrow (\exists V2x \in A.27b. ((ap\ (ap\ (c.2Epair\_2E\_2C \\ & A.27a\ 2)\ V1v)\ c.2Ebool\_2ET) = (ap\ V0f\ V2x)))))) \end{aligned} \quad (47)$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\
& \quad \forall V0f \in (A\_27b^{A\_27a}). (\forall V1s \in (2^{A\_27a}). (\forall V2t \in \\
& \quad (2^{A\_27b}). ((p\ (ap\ (ap\ (ap\ (c\_2Epred\_set\_2EINJ\ A\_27a\ A\_27b)\ V0f)\ \\
& \quad V1s)\ V2t)) \Leftrightarrow ((\forall V3x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ \\
& \quad V3x)\ V1s)) \Rightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27b)\ (ap\ V0f\ V3x))\ V2t)))))) \wedge \\
& \quad (\forall V4x \in A\_27a. (\forall V5y \in A\_27a. (((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ \\
& \quad A\_27a)\ V4x)\ V1s)) \wedge (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V5y)\ V1s))) \Rightarrow \\
& \quad (((ap\ V0f\ V4x) = (ap\ V0f\ V5y)) \Leftrightarrow (V4x = V5y))))))))) \\
& \hspace{15em} (48)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\
& \quad \forall V0f \in (A\_27b^{A\_27a}). (\forall V1s \in (2^{A\_27a}). (\forall V2t \in \\
& \quad (2^{A\_27b}). ((p\ (ap\ (ap\ (ap\ (c\_2Epred\_set\_2EBIJ\ A\_27a\ A\_27b)\ V0f)\ \\
& \quad V1s)\ V2t)) \Leftrightarrow ((\forall V3x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ \\
& \quad V3x)\ V1s)) \Rightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27b)\ (ap\ V0f\ V3x))\ V2t)))))) \wedge \\
& \quad (\exists V4g \in (A\_27a^{A\_27b}). ((\forall V5x \in A\_27b. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ \\
& \quad A\_27b)\ V5x)\ V2t)) \Rightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ (ap\ V4g\ V5x))\ \\
& \quad V1s)))))) \wedge ((\forall V6x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ \\
& \quad V6x)\ V1s)) \Rightarrow ((ap\ V4g\ (ap\ V0f\ V6x)) = V6x))) \wedge (\forall V7x \in A\_27b. ( \\
& \quad (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27b)\ V7x)\ V2t)) \Rightarrow ((ap\ V0f\ (ap\ V4g\ V7x)) = \\
& \quad V7x))))))))) \\
& \hspace{15em} (49)
\end{aligned}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (50)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (51)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\
((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \quad (52)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\
((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))) \quad (53)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (54)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \Leftrightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee ((p \ V1q) \vee (p \ V2r))) \wedge (((p \ V0p) \vee ((\neg \\
& p \ V2r)) \vee (\neg(p \ V1q)))) \wedge (((p \ V1q) \vee ((\neg(p \ V2r)) \vee (\neg(p \ V0p)))) \wedge ((p \ V2r) \vee \\
& ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{55}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \wedge (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee ((\neg(p \ V1q)) \vee (\neg(p \ V2r)))) \wedge (((p \ V1q) \vee \\
& (\neg(p \ V0p))) \wedge ((p \ V2r) \vee (\neg(p \ V0p)))))))))
\end{aligned} \tag{56}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \vee (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (\neg(p \ V1q))) \wedge (((p \ V0p) \vee (\neg(p \ V2r))) \wedge \\
& ((p \ V1q) \vee ((p \ V2r) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{57}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \ V0p) \Leftrightarrow ( \\
& (p \ V1q) \Rightarrow (p \ V2r))) \Leftrightarrow (((p \ V0p) \vee (p \ V1q)) \wedge (((p \ V0p) \vee (\neg(p \ V2r))) \wedge (( \\
& \neg(p \ V1q)) \vee ((p \ V2r) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{58}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p \ V0p) \Leftrightarrow (\neg(p \ V1q))) \Leftrightarrow (((p \ V0p) \vee \\
& (p \ V1q)) \wedge ((\neg(p \ V1q)) \vee (\neg(p \ V0p))))))
\end{aligned} \tag{59}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (\forall V3s \in \\
& 2. (((p \ V0p) \Leftrightarrow (ap \ (ap \ (ap \ (c.2Ebool.2ECOND \ 2) \ V1q) \ V2r) \ V3s))) \Leftrightarrow \\
& (((p \ V0p) \vee ((p \ V1q) \vee (\neg(p \ V3s)))) \wedge (((p \ V0p) \vee ((\neg(p \ V2r)) \vee (\neg(p \ V1q)))) \wedge \\
& (((p \ V0p) \vee ((\neg(p \ V2r)) \vee (\neg(p \ V3s)))) \wedge (((\neg(p \ V1q)) \vee ((p \ V2r) \vee (\neg \\
& p \ V0p)))) \wedge ((p \ V1q) \vee ((p \ V3s) \vee (\neg(p \ V0p))))))))))
\end{aligned} \tag{60}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow (p \ V0p))) \tag{61}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \Rightarrow (p \ V1q))) \Rightarrow (\neg(p \ V1q)))) \tag{62}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \ V0p) \vee (p \ V1q))) \Rightarrow (\neg(p \ V0p)))) \tag{63}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))))) \quad (64)$$

Assume the following.

$$(\forall V0p \in 2.((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \quad (65)$$

**Theorem 1**

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0A \in (2^{A\_27a}).(\forall V1B \in (2^{A\_27b}).((p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardeq \\ A\_27a\ A\_27b)\ V0A)\ V1B)) \Rightarrow (p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardeq\ ((ty\_2Eoption\_2Eoption \\ A\_27a)^{A\_27a})\ ((ty\_2Eoption\_2Eoption\ A\_27b)^{A\_27b}))\ (ap\ (c\_2Ecardinal\_2Ebijns \\ A\_27a)\ V0A))\ (ap\ (c\_2Ecardinal\_2Ebijns\ A\_27b)\ V1B)))))) \end{aligned}$$