

# thm\_2Ecardinal\_2Eset\_\_exp\_\_cardle\_\_cong (TMdiVKkY2HuzAmXdE5gfatFgbFC5jn7Dtws)

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**Definition 1** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p \text{ (ap } P \ x)) \text{ then (the } (\lambda x. x \in A \wedge p \ x)) \text{ of type } \iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj\_o } (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define `c_2Ebool_2ET` to be  $(\text{ap } (\text{ap } (\text{c_2Emin_2E_3D } (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 4** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } (\text{ap } (\text{c_2Emin_2E_3D } (2^{A-27a}))))$

**Definition 5** We define `c_2Ebool_2EF` to be  $(\text{ap } (\text{c_2Ebool_2E_21 } 2) (\lambda V0t \in 2.V0t))$ .

**Definition 6** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. \text{inj\_o } (p \Rightarrow q)$  of type  $\iota$ .

**Definition 7** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (\text{ap } (\text{c_2Ebool_2E_21 } 2) (\lambda V2t \in 2.V2t))))$

**Definition 8** We define `c_2Ebool_2ECOND` to be  $\lambda A. 27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A. 27a. (\lambda V2t2 \in A. 27a. (\text{ap } (\text{c_2Emin_2E_3D } (2^{A-27a}))))))$

**Definition 9** We define `c_2Ebool_2EIN` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. (\lambda V1f \in (2^{A-27a}). (\text{ap } V1f \ V0x)))$

**Definition 10** We define `c_2Epred__set_2EINJ` to be  $\lambda A. 27a : \iota. \lambda A. 27b : \iota. \lambda V0f \in (A. 27b^{A-27a}). \lambda V1s \in (2^{A-27a})$

**Definition 11** We define `c_2Ebool_2E_3F` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } V0P \ (\text{ap } (\text{c_2Emin_2E_40 } (2^{A-27a}))))$

**Definition 12** We define `c_2Ecardinal_2Ecardleq` to be  $\lambda A. 27a : \iota. \lambda A. 27b : \iota. \lambda V0s1 \in (2^{A-27a}). \lambda V1s2 \in (2^{A-27a})$

Let `ty_2Eone_2Eone` :  $\iota$  be given. Assume the following.

$$\text{nonempty } \text{ty\_2Eone\_2Eone} \tag{1}$$

**Definition 13** We define `c_2Eone_2Eone` to be  $(\text{ap } (\text{c_2Emin_2E_40 } \text{ty\_2Eone\_2Eone}) (\lambda V0x \in \text{ty\_2Eone\_2Eone}))$

**Definition 14** We define `c_2Ebool_2E_7E` to be  $(\lambda V0t \in 2. (\text{ap } (\text{ap } (\text{c_2Emin_2E_3D_3D_3E } V0t) \ \text{c_2Ebool_2E_7E})))$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \quad (2)$$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \quad (3)$$

**Definition 15** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap\ (c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b)\ V0e)$

Let  $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Eoption\_2Eoption\ A0) \quad (4)$$

Let  $c\_2Eoption\_2Eoption\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eoption\_2Eoption\_ABS\ A\_27a \in ((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ ty\_2Eone\_2Eone)}) \quad (5)$$

**Definition 16** We define  $c\_2Eoption\_2ENONE$  to be  $\lambda A\_27a : \iota.(ap\ (c\_2Eoption\_2Eoption\_ABS\ A\_27a)\ 0)$

**Definition 17** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap\ (c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b)\ V0e)$

**Definition 18** We define  $c\_2Eoption\_2ESOME$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.(ap\ (c\_2Eoption\_2Eoption\_ABS\ A\_27a)\ V0x)$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (6)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (7)$$

**Definition 19** We define  $c\_2Epair\_2E2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b)\ (V0x, V1y))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}}) \quad (8)$$

**Definition 20** We define  $c\_2Ecardinal\_2Eset\_exp$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0A \in (2^{A\_27b}).\lambda V1B \in (2^{A\_27a}).(ap\ (c\_2Epair\_2E2C\ A\_27a\ A\_27b)\ (V0A, V1B))$

Let  $c\_2Eoption\_2EOPTION\_MAP : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Eoption\_2EOPTION\_MAP \\ A\_27a\ A\_27b \in (((ty\_2Eoption\_2Eoption\ A\_27b)^{(ty\_2Eoption\_2Eoption\ A\_27a)})^{(A\_27b^{A\_27a}}) \end{aligned} \quad (9)$$

**Definition 21** We define  $c\_2Eoption\_2Esome$  to be  $\lambda A\_27a : \iota. \lambda V0P \in (2^{A\_27a}). (ap\ (ap\ (ap\ (c\_2Ebool\_2Esome\ A\_27a\ P))))$

Let  $c\_2Eoption\_2Eoption\_CASE : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Eoption\_2Eoption\_CASE \\ A\_27a\ A\_27b \in (((A\_27b^{(A\_27b^{A\_27a}})})^{(ty\_2Eoption\_2Eoption\ A\_27a)}) \end{aligned} \quad (10)$$

Let  $c\_2Epred\_set\_2ECHOICE : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epred\_set\_2ECHOICE\ A\_27a \in \\ (A\_27a^{(2^{A\_27a})}) \end{aligned} \quad (11)$$

**Definition 22** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota. (\lambda V0x \in A\_27a. c\_2Ebool\_2Esome\ A\_27a\ x)$ .

**Definition 23** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c\_2Ebool\_2E\_21\ 2)\ (V0t1\ V1t2))))$

Assume the following.

$$True \quad (12)$$

Assume the following.

$$\begin{aligned} (\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p \\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \end{aligned} \quad (13)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p\ V0t))) \quad (14)$$

Assume the following.

$$(\forall V0t \in 2. ((p\ V0t) \vee \neg(p\ V0t))) \quad (15)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in \\ A\_27a. (p\ V0t)) \Leftrightarrow (p\ V0t))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2. ((\exists V1x \in \\ A\_27a. (p\ V0t)) \Leftrightarrow (p\ V0t))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2. (((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow \\ (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \\ (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \vee (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \vee True) \Leftrightarrow True) \wedge \\
& (((False \vee (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee False) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee \\
& (p \ V0t)) \Leftrightarrow (p \ V0t))))))
\end{aligned} \tag{19}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Rightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Rightarrow True) \Leftrightarrow \\
& True) \wedge (((False \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (( \\
& (p \ V0t) \Rightarrow False) \Leftrightarrow (\neg(p \ V0t))))))
\end{aligned} \tag{20}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0t \in 2.((\neg(\neg(p \ V0t))) \Leftrightarrow (p \ V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge \\
& ((\neg False) \Leftrightarrow True)))
\end{aligned} \tag{21}$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.(V0x = V0x)) \tag{22}$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{23}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in \\
& A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x))))
\end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow \forall A\_27b.nonempty \ A\_27b \Rightarrow ( \\
& \forall V0f \in (A\_27b^{A\_27a}).(\forall V1g \in (A\_27b^{A\_27a}).((V0f = \\
& V1g) \Leftrightarrow (\forall V2x \in A\_27a.((ap \ V0f \ V2x) = (ap \ V1g \ V2x))))))
\end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow (\neg(p \ V0t))) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\
& p \ V0t))))))
\end{aligned} \tag{26}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\forall V1x \in \\
& A\_27a.(p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\exists V2x \in A\_27a.(\neg(p \ (ap \ V0P \ V2x))))))
\end{aligned} \tag{27}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\exists V1x \in \\
& A\_27a.(p \ (ap \ V0P \ V1x)))) \Leftrightarrow (\forall V2x \in A\_27a.(\neg(p \ (ap \ V0P \ V2x))))))
\end{aligned} \tag{28}$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in \\ & (2^{A.27a}).((\forall V2x \in A.27a.((p (ap V0P V2x)) \wedge (p (ap V1Q V2x)))) \Leftrightarrow \\ & ((\forall V3x \in A.27a.(p (ap V0P V3x))) \wedge (\forall V4x \in A.27a.(p ( \\ & \quad ap V1Q V4x))))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).((p V0P) \wedge (\forall V2x \in A.27a.(p (ap V1Q V2x)))) \Leftrightarrow (\forall V3x \in \\ & \quad A.27a.((p V0P) \wedge (p (ap V1Q V3x)))))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in \\ & (2^{A.27a}).((\exists V2x \in A.27a.((p (ap V0P V2x)) \vee (p (ap V1Q V2x)))) \Leftrightarrow \\ & ((\exists V3x \in A.27a.(p (ap V0P V3x))) \vee (\exists V4x \in A.27a.(p ( \\ & \quad ap V1Q V4x)))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).((\exists V2x \in A.27a.((p V0P) \wedge (p (ap V1Q V2x)))) \Leftrightarrow ((p \\ & \quad V0P) \wedge (\exists V3x \in A.27a.(p (ap V1Q V3x)))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).((\forall V2x \in A.27a.((p V0P) \vee (p (ap V1Q V2x)))) \Leftrightarrow ((p \\ & \quad V0P) \vee (\forall V3x \in A.27a.(p (ap V1Q V3x)))))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg( \\ & p V0A)) \vee (\neg(p V1B)))))) \wedge (((\neg((p V0A) \vee (p V1B))) \Leftrightarrow ((\neg(p V0A)) \wedge (\neg(p V1B)))))) \end{aligned} \quad (34)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V0A) \wedge ( \\ & (p V1B) \vee (p V2C))) \Leftrightarrow (((p V0A) \wedge (p V1B)) \vee ((p V0A) \wedge (p V2C)))))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V0A) \vee ( \\ & (p V1B) \wedge (p V2C))) \Leftrightarrow (((p V0A) \vee (p V1B)) \wedge ((p V0A) \vee (p V2C)))))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V1B) \wedge \\ & (p V2C)) \vee (p V0A)) \Leftrightarrow (((p V1B) \vee (p V0A)) \wedge ((p V2C) \vee (p V0A)))))) \end{aligned} \quad (37)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (38)$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x_{.27} \in 2.(\forall V2y \in 2.(\forall V3y_{.27} \in 2.(((p V0x) \Leftrightarrow (p V1x_{.27})) \wedge ((p V1x_{.27}) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_{.27})))))) \Rightarrow ((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_{.27}) \Rightarrow (p V3y_{.27})))))) \quad (39)$$

Assume the following.

$$\begin{aligned} \forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow & (\forall V0P \in 2.(\forall V1Q \in 2. \\ & (\forall V2x \in A_{.27a}.(\forall V3x_{.27} \in A_{.27a}.(\forall V4y \in A_{.27a}. \\ & (\forall V5y_{.27} \in A_{.27a}.(((p V0P) \Leftrightarrow (p V1Q)) \wedge ((p V1Q) \Rightarrow (V2x = V3x_{.27})) \wedge \\ & ((\neg(p V1Q)) \Rightarrow (V4y = V5y_{.27})))) \Rightarrow ((ap (ap (ap (c_{.2Ebool_{.2ECOND}} \ A_{.27a}) \\ & V0P) V2x) V4y) = (ap (ap (ap (c_{.2Ebool_{.2ECOND}} \ A_{.27a}) V1Q) V3x_{.27}) \\ & V5y_{.27})))))))))) \end{aligned} \quad (40)$$

Assume the following.

$$\forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow (\forall V0P \in (2^{A_{.27a}}).(\forall V1a \in A_{.27a}.((\exists V2x \in A_{.27a}.((V2x = V1a) \wedge (p (ap V0P V2x)))) \Leftrightarrow (p (ap V0P V1a)))))) \quad (41)$$

Assume the following.

$$\begin{aligned} \forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow & ((\forall V0t1 \in A_{.27a}.(\forall V1t2 \in \\ & A_{.27a}.((ap (ap (ap (c_{.2Ebool_{.2ECOND}} \ A_{.27a}) \ c_{.2Ebool_{.2ET}}) V0t1) \\ & V1t2) = V0t1))) \wedge (\forall V2t1 \in A_{.27a}.(\forall V3t2 \in A_{.27a}.((ap \\ & (ap (ap (c_{.2Ebool_{.2ECOND}} \ A_{.27a}) \ c_{.2Ebool_{.2EF}}) V2t1) V3t2) = V3t2)))))) \end{aligned} \quad (42)$$

Assume the following.

$$\begin{aligned} \forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow & \forall A_{.27b}.nonempty \ A_{.27b} \Rightarrow ( \\ & (\forall V0v \in A_{.27b}.(\forall V1f \in (A_{.27b}^{A_{.27a}}).((ap (ap (ap (c_{.2Eoption_{.2Eoption\_CASE}} \ A_{.27a} \ A_{.27b}) (c_{.2Eoption_{.2ENONE}} \ A_{.27a}) V0v) V1f) = V0v))) \wedge (\forall V2x \in \\ & A_{.27a}.(\forall V3v \in A_{.27b}.(\forall V4f \in (A_{.27b}^{A_{.27a}}).((ap (ap \\ & (ap (c_{.2Eoption_{.2Eoption\_CASE}} \ A_{.27a} \ A_{.27b}) (ap (c_{.2Eoption_{.2ESOME}} \ A_{.27a}) V2x) V3v) V4f) = (ap V4f V2x))))))))) \end{aligned} \quad (43)$$

Assume the following.

$$\forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow (\forall V0x \in A_{.27a}.(\forall V1y \in A_{.27a}.(((ap (c_{.2Eoption_{.2ESOME}} \ A_{.27a}) V0x) = (ap (c_{.2Eoption_{.2ESOME}} \ A_{.27a}) V1y)) \Leftrightarrow (V0x = V1y)))))) \quad (44)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\neg((c\_2Eoption\_2ENONE\ A\_27a) = (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V0x)))) \quad (45)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow & ( \\ (\forall V0f \in (A\_27b^{A\_27a}). (\forall V1x \in A\_27a. ((ap\ (ap\ (c\_2Eoption\_2EOPTION\_MAP & \\ A\_27a\ A\_27b)\ V0f)\ (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V1x)) = (ap\ (c\_2Eoption\_2ESOME & \\ A\_27b)\ (ap\ V0f\ V1x)))))) \wedge (\forall V2f \in (A\_27b^{A\_27a}). ((ap\ (ap\ (c\_2Eoption\_2EOPTION\_MAP & \\ A\_27a\ A\_27b)\ V2f)\ (c\_2Eoption\_2ENONE\ A\_27a)) = (c\_2Eoption\_2ENONE & \\ A\_27b)))))) & \quad (46) \end{aligned}$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow & ( \\ \forall V0f \in (A\_27b^{A\_27a}). (\forall V1x \in (ty\_2Eoption\_2Eoption & \\ A\_27a)\ (ap\ V2y \in A\_27b. (((ap\ (ap\ (c\_2Eoption\_2EOPTION\_MAP & \\ A\_27a\ A\_27b)\ V0f)\ V1x) = (ap\ (c\_2Eoption\_2ESOME\ A\_27b)\ V2y)) \Leftrightarrow (\exists V3z \in & \\ A\_27a. ((V1x = (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V3z)) \wedge (V2y = (ap\ V0f & \\ V3z)))))))))) & \quad (47) \end{aligned}$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}). (\forall V1Q \in & \\ (2^{(ty\_2Eoption\_2Eoption\ A\_27a)}). (((\forall V2x \in A\_27a. ((p & \\ (ap\ V0P\ V2x)) \Rightarrow (p\ (ap\ V1Q\ (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V2x)))))) \wedge & \\ ((\forall V3x \in A\_27a. (\neg(p\ (ap\ V0P\ V3x)))) \Rightarrow (p\ (ap\ V1Q\ (c\_2Eoption\_2ENONE & \\ A\_27a)))))) \Rightarrow (p\ (ap\ V1Q\ (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V0P)))))) & \quad (48) \end{aligned}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((ap\ (c\_2Eoption\_2ESOME\ A\_27a) & \\ (\lambda V0x \in A\_27a.c\_2Ebool\_2EF)) = (c\_2Eoption\_2ENONE\ A\_27a)) \quad (49)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow & ( \\ \forall V0opt \in (ty\_2Eoption\_2Eoption\ A\_27a). (\forall V1nc \in A\_27b. & \\ (\forall V2sc \in (A\_27b^{A\_27a}). (\forall V3v \in A\_27b. (((ap\ (ap\ (ap & \\ (c\_2Eoption\_2Eoption\_CASE\ A\_27a\ A\_27b)\ V0opt)\ V1nc)\ V2sc) = V3v) \Leftrightarrow & \\ (((V0opt = (c\_2Eoption\_2ENONE\ A\_27a)) \wedge (V1nc = V3v)) \vee (\exists V4x \in & \\ A\_27a. ((V0opt = (ap\ (c\_2Eoption\_2ESOME\ A\_27a)\ V4x)) \wedge ((ap\ V2sc & \\ V4x) = V3v)))))))))) & \quad (50) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0x \in A\_27a. (\forall V1y \in A\_27b. (\forall V2a \in A\_27a. (\forall V3b \in \\ & \quad A\_27b. (((ap\ (ap\ (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V0x)\ V1y) = (ap\ (ap \\ & \quad (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b))))))))) \end{aligned} \quad (51)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in ((ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}). (\forall V1v \in \\ & \quad A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V1v)\ (ap\ (c\_2Epred\_set\_2EGSPEC \\ & \quad A\_27a\ A\_27b)\ V0f))) \Leftrightarrow (\exists V2x \in A\_27b. ((ap\ (ap\ (c\_2Epair\_2E\_2C \\ & \quad A\_27a\ 2)\ V1v)\ c\_2Ebool\_2ET) = (ap\ V0f\ V2x))))))))) \end{aligned} \quad (52)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\neg (p\ (ap\ (ap \\ & \quad (c\_2Ebool\_2EIN\ A\_27a)\ V0x)\ (c\_2Epred\_set\_2EEMPTY\ A\_27a)))))) \end{aligned} \quad (53)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). ((\neg (V0s = \\ & \quad (c\_2Epred\_set\_2EEMPTY\ A\_27a))) \Rightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a) \\ & \quad (ap\ (c\_2Epred\_set\_2ECHOICE\ A\_27a)\ V0s))\ V0s)))))) \end{aligned} \quad (54)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in (A\_27b^{A\_27a}). ((\forall V1s \in (2^{A\_27b}). (p\ (ap\ (ap \\ & \quad (ap\ (c\_2Epred\_set\_2EINJ\ A\_27a\ A\_27b)\ V0f)\ (c\_2Epred\_set\_2EEMPTY \\ & \quad A\_27a)\ V1s))) \wedge (\forall V2s \in (2^{A\_27a}). ((p\ (ap\ (ap\ (ap\ (c\_2Epred\_set\_2EINJ \\ & \quad A\_27a\ A\_27b)\ V0f)\ V2s)\ (c\_2Epred\_set\_2EEMPTY\ A\_27b)))) \Leftrightarrow (V2s = \\ & \quad (c\_2Epred\_set\_2EEMPTY\ A\_27a))))))))) \end{aligned} \quad (55)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg (\neg (p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (56)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg (p\ V0A)) \Rightarrow False))) \quad (57)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg ((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & \quad (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg (p\ V1B)) \Rightarrow False)))))) \end{aligned} \quad (58)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (((\neg ((\neg (p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\ & \quad ((p\ V0A) \Rightarrow ((\neg (p\ V1B)) \Rightarrow False)))))) \end{aligned} \quad (59)$$



Assume the following.

$$(\forall V0A \in 2.((\neg(p V0A)) \Rightarrow False) \Rightarrow ((p V0A) \Rightarrow False) \Rightarrow False)) \quad (60)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee (\neg( \\ & p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\ & ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \end{aligned} \quad (61)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee \\ & (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p)))))) \end{aligned} \quad (62)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge \\ & ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))) \end{aligned} \quad (63)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge (( \\ & \neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))) \end{aligned} \quad (64)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\ & (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p)))))) \end{aligned} \quad (65)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p))) \quad (66)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q)))) \quad (67)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V0p)))) \quad (68)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))) \quad (69)$$

Assume the following.

$$(\forall V0p \in 2.((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \quad (70)$$

**Theorem 1**

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c. \\ & nonempty\ A\_27c \Rightarrow \forall A\_27d.nonempty\ A\_27d \Rightarrow (\forall V0b \in (2^{A\_27a}). \\ & (\forall V1d \in (2^{A\_27b}). (\forall V2a \in (2^{A\_27c}). (\forall V3c \in \\ (2^{A\_27d}). (((V0b = (c\_2Epred\_set\_2EEMPTY\ A\_27a)) \Rightarrow (V1d = (c\_2Epred\_set\_2EEMPTY \\ & A\_27b))) \Rightarrow (((p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardleq\ A\_27c\ A\_27a)\ V2a) \\ & V0b)) \wedge (p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardleq\ A\_27d\ A\_27b)\ V3c)\ V1d))) \Rightarrow \\ & (p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardleq\ ((ty\_2Eoption\_2Eoption\ A\_27c)^{A\_27d}) \\ & ((ty\_2Eoption\_2Eoption\ A\_27a)^{A\_27b}))\ (ap\ (ap\ (c\_2Ecardinal\_2Eset\_exp \\ & A\_27d\ A\_27c)\ V2a)\ V3c))\ (ap\ (ap\ (c\_2Ecardinal\_2Eset\_exp\ A\_27b \\ & A\_27a)\ V0b)\ V1d)))))))))) \end{aligned}$$