

thm\_2Ecomparison\_2Egood\_\_cmp\_\_Less\_\_irrefl\_\_trans  
 (TM-  
 Roz4Jq7wKCDEPDCxnwDSHViHo5WqyZyfE)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A. \lambda a : \iota. (\lambda V0P \in (2^{A-27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A-27a})) (\lambda V1x \in 2.V1x)) (\lambda V2x \in 2.V2x)))$

**Definition 4** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2. inj\_o (V0t1 = V1t2))))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (1)$$

Let  $ty\_2EternaryComparisons\_2Eordering : \iota$  be given. Assume the following.

$$nonempty\ ty\_2EternaryComparisons\_2Eordering \quad (2)$$

Let  $c\_2EternaryComparisons\_2Eordering2num : \iota$  be given. Assume the following.

$$c\_2EternaryComparisons\_2Eordering2num \in (ty\_2Enum\_2Enum^{ty\_2EternaryComparisons\_2Eordering}) \quad (3)$$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in omega \quad (4)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{omega}) \quad (5)$$

**Definition 7** We define  $c\_2EEnum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

**Definition 8** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREPE\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (6)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^\omega) \quad (7)$$

**Definition 9** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num\ ($

Let  $c : \mathbb{E}arithmic \mathbb{E}B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (8)$$

**Definition 10** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic\ 2EBIT1\ n)\ V)$

**Definition 11** We define `c_2Earithmetic_2ENUMERAL` to be  $\lambda V0x \in ty\_2Enum\_2Enum. V0x$ .

**Definition 12** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

**Definition 13** We define  $c_{\text{Emin}}.40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p \text{ (ap } P \text{ } x)) \text{ then } (\lambda x.x \in A \wedge \text{of type } \iota \Rightarrow \iota)$ .

**Definition 14** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A.27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A.27a.(\lambda V2t2 \in A.27a.($

**Definition 15** We define  $c_2Eb0o\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap\ (ap\ c_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c_2Eb0o\_2E\_7E))$

**Definition 16** We define  $c_2Ebool\_2E\_3F$  to be  $\lambda A.\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c_2Emin\_2E\ 40$

**Definition 17** We define  $c_2\text{Eprim\_rec}_2\text{E\_3C}$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.$

**Definition 18** We define  $\text{c\_2EternaryComparisons\_2Ordering\_CASE}$  to be  $\lambda A. \exists g : \lambda V. 0x \in tu \_2EternaryComparisons$

Let  $c$  2EternaryComparisons 2EGREATER :  $t$  be given. Assume the follow-

ing.

<sup>9</sup> *c\_2EternaryComparisons\_2EGREATER*  $\in$  *ty\_2EternaryComparisons\_2Ordering*

Let  $c_{\text{2EternaryComparisons\_2EQUAL}} : \iota$  be given. Assume the following.

$c\_2EternaryComparisons\_2EQUAL \in ty\_2EternaryComparisons\_2Ordering$  (10)

Let  $c_{\text{2EternaryComparisons\_2ELLESS}} : \iota$  be given. Assume the following.

$$c_{\text{2EternaryComparisons\_2ELESS}} \in \text{ty\_2EternaryComparisons\_2Eordering} \quad (11)$$

**Definition 19** We define  $c\_2Erelation\_2Etransitive$  to be  $\lambda A\_27a : \iota. \lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap\ (c\_2Ebool\_2Etransitive\ A\_27a)\ V0R)$ .

**Definition 20** We define  $c\_2Ecomparison\_2Egood\_cmp$  to be  $\lambda A.27a : \iota.\lambda V0cmp \in ((ty\_2EternaryComparison \wedge V0 \neq 0) \rightarrow V1)$

**Definition 21** We define  $c\_2Erelation\_2Eirreflexive$  to be  $\lambda A.27a : \iota.\lambda V0R \in ((2^{A\_27a})^A)^{A\_27a}.$ ( $ap\ (c\_2Ebool\_2Eirreflexive\ A)\ R$ )

Assume the following.

True (12)

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. ((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (13)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\;V0t))) \quad (14)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p \ V0t)) \Leftrightarrow (p \ V0t))) \quad (15)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p \vee 0t)) \Leftrightarrow (p \vee 0t)) \wedge (((p \vee 0t) \wedge True) \Leftrightarrow (p \vee 0t)) \wedge (((False \wedge (p \vee 0t)) \Leftrightarrow False) \wedge (((p \vee 0t) \wedge False) \Leftrightarrow False) \wedge (((p \vee 0t) \wedge (p \vee 0t)) \Leftrightarrow (p \vee 0t))))))) \quad (16)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t))))))) \quad (17)$$

Assume the following.

$$((\forall V \forall t \in 2. ((\neg(\neg(p V 0t))) \Leftrightarrow (p V 0t))) \wedge (((\neg \text{True}) \Leftrightarrow \text{False}) \wedge ((\neg \text{False}) \Leftrightarrow \text{True}))) \quad (18)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (19)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow (\neg(p \ V0t))) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p \ V0t))))))) \quad (20)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (21)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in 2. (\forall V1x\_27 \in 2. (\forall V2y \in 2. (\forall V3y\_27 \in \\ & 2. (((((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))))) \Rightarrow \\ & (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27))))))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned}
& \forall A_{.27a}. nonempty A_{.27a} \Rightarrow (((\neg(c_{.2}EternaryComparisons_{.2}ELESS = \\
& c_{.2}EternaryComparisons_{.2}EEQUAL)) \wedge ((\neg(c_{.2}EternaryComparisons_{.2}ELESS = \\
& c_{.2}EternaryComparisons_{.2}EGREATER)) \wedge (\neg(c_{.2}EternaryComparisons_{.2}EEQUAL = \\
& c_{.2}EternaryComparisons_{.2}EGREATER)))) \wedge ((\forall V0v0 \in A_{.27a}. \\
& (\forall V1v1 \in A_{.27a}. (\forall V2v2 \in A_{.27a}. ((ap (ap (ap (c_{.2}EternaryComparisons_{.2}Eordering_{.2}CASE \\
& A_{.27a}) c_{.2}EternaryComparisons_{.2}ELESS) V0v0) V1v1) V2v2) = V0v0)))) \wedge \\
& ((\forall V3v0 \in A_{.27a}. (\forall V4v1 \in A_{.27a}. (\forall V5v2 \in A_{.27a}. \\
& ((ap (ap (ap (c_{.2}EternaryComparisons_{.2}Eordering_{.2}CASE A_{.27a}) \\
& c_{.2}EternaryComparisons_{.2}EEQUAL) V3v0) V4v1) V5v2) = V4v1)))) \wedge \\
& (\forall V6v0 \in A_{.27a}. (\forall V7v1 \in A_{.27a}. (\forall V8v2 \in A_{.27a}. \\
& ((ap (ap (ap (c_{.2}EternaryComparisons_{.2}Eordering_{.2}CASE A_{.27a}) \\
& c_{.2}EternaryComparisons_{.2}EGREATER) V6v0) V7v1) V8v2) = V8v2)))) \wedge \\
& ((\forall V9a \in ty_{.2}EternaryComparisons_{.2}Eordering. ((V9a = c_{.2}EternaryComparisons_{.2}ELESS) \vee \\
& (V9a = c_{.2}EternaryComparisons_{.2}EEQUAL) \vee (V9a = c_{.2}EternaryComparisons_{.2}EGREATER))) \wedge \\
& (\forall V10cmp \in ((ty_{.2}EternaryComparisons_{.2}Eordering^{A_{.27a}})^{A_{.27a}}). \\
& ((p (ap (c_{.2}Ecomparison_{.2}Egood_{.2}cmp A_{.27a}) V10cmp)) \Leftrightarrow ((\forall V11x \in \\
& A_{.27a}. ((ap (ap V10cmp V11x) V11x) = c_{.2}EternaryComparisons_{.2}EEQUAL)) \wedge \\
& ((\forall V12x \in A_{.27a}. (\forall V13y \in A_{.27a}. ((ap (ap V10cmp V12x) \\
& V13y) = c_{.2}EternaryComparisons_{.2}EEQUAL) \Rightarrow ((ap (ap V10cmp V13y) \\
& V12x) = c_{.2}EternaryComparisons_{.2}EEQUAL)))) \wedge ((\forall V14x \in \\
& A_{.27a}. (\forall V15y \in A_{.27a}. ((ap (ap V10cmp V14x) V15y) = c_{.2}EternaryComparisons_{.2}EGREATER) \Leftrightarrow \\
& ((ap (ap V10cmp V15y) V14x) = c_{.2}EternaryComparisons_{.2}ELESS)))) \wedge \\
& ((\forall V16x \in A_{.27a}. (\forall V17y \in A_{.27a}. (\forall V18z \in A_{.27a}. \\
& (((ap (ap V10cmp V16x) V17y) = c_{.2}EternaryComparisons_{.2}EEQUAL) \wedge \\
& ((ap (ap V10cmp V17y) V18z) = c_{.2}EternaryComparisons_{.2}ELESS)) \Rightarrow \\
& ((ap (ap V10cmp V16x) V18z) = c_{.2}EternaryComparisons_{.2}ELESS)))) \wedge \\
& ((\forall V19x \in A_{.27a}. (\forall V20y \in A_{.27a}. (\forall V21z \in A_{.27a}. \\
& (((ap (ap V10cmp V19x) V20y) = c_{.2}EternaryComparisons_{.2}ELESS) \wedge \\
& ((ap (ap V10cmp V20y) V21z) = c_{.2}EternaryComparisons_{.2}EEQUAL)) \Rightarrow \\
& ((ap (ap V10cmp V19x) V21z) = c_{.2}EternaryComparisons_{.2}ELESS)))) \wedge \\
& ((\forall V22x \in A_{.27a}. (\forall V23y \in A_{.27a}. (\forall V24z \in A_{.27a}. \\
& (((ap (ap V10cmp V22x) V23y) = c_{.2}EternaryComparisons_{.2}EEQUAL) \wedge \\
& ((ap (ap V10cmp V23y) V24z) = c_{.2}EternaryComparisons_{.2}EEQUAL)) \Rightarrow \\
& ((ap (ap V10cmp V22x) V24z) = c_{.2}EternaryComparisons_{.2}EEQUAL)))) \wedge \\
& (\forall V25x \in A_{.27a}. (\forall V26y \in A_{.27a}. (\forall V27z \in A_{.27a}. \\
& (((ap (ap V10cmp V25x) V26y) = c_{.2}EternaryComparisons_{.2}ELESS) \wedge \\
& ((ap (ap V10cmp V26y) V27z) = c_{.2}EternaryComparisons_{.2}ELESS)) \Rightarrow \\
& ((ap (ap V10cmp V25x) V27z) = c_{.2}EternaryComparisons_{.2}ELESS))))))))))) \\
& (23)
\end{aligned}$$

Assume the following.

$$\begin{aligned} \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0cmp \in ((\text{ty\_2EternaryComparisons\_2Eordering}^{A\_27a})^{A\_27a}). \\ ((p (\text{ap} (\text{c\_2Ecomparison\_2Egood\_cmp } A\_27a) V0cmp)) \Rightarrow (p (\text{ap} (\text{c\_2Erelation\_2Etransitive} \\ A\_27a) (\lambda V1k \in A\_27a.(\lambda V2k\_27 \in A\_27a.(\text{ap} (\text{ap} (\text{c\_2Emin\_2E\_3D} \\ \text{ty\_2EternaryComparisons\_2Eordering}) (\text{ap} (\text{ap} V0cmp V1k) V2k\_27)) \\ c\_2EternaryComparisons\_2ELESS))))))) \\ (24) \end{aligned}$$

### Theorem 1

$$\begin{aligned} \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0cmp \in ((\text{ty\_2EternaryComparisons\_2Eordering}^{A\_27a})^{A\_27a}). \\ ((p (\text{ap} (\text{c\_2Ecomparison\_2Egood\_cmp } A\_27a) V0cmp)) \Rightarrow ((p (\text{ap} (\text{c\_2Erelation\_2Eirreflexive} \\ A\_27a) (\lambda V1k \in A\_27a.(\lambda V2k\_27 \in A\_27a.(\text{ap} (\text{ap} (\text{c\_2Emin\_2E\_3D} \\ \text{ty\_2EternaryComparisons\_2Eordering}) (\text{ap} (\text{ap} V0cmp V1k) V2k\_27)) \\ c\_2EternaryComparisons\_2ELESS))))))) \wedge (p (\text{ap} (\text{c\_2Erelation\_2Etransitive} \\ A\_27a) (\lambda V3k \in A\_27a.(\lambda V4k\_27 \in A\_27a.(\text{ap} (\text{ap} (\text{c\_2Emin\_2E\_3D} \\ \text{ty\_2EternaryComparisons\_2Eordering}) (\text{ap} (\text{ap} V0cmp V3k) V4k\_27)) \\ c\_2EternaryComparisons\_2ELESS))))))) \\ \end{aligned}$$