

thm_2Ecomparison_2Epair__cmp__lexTO (TMd1eb8sTGAu5rEZrg2fePDTH5Vv51vjkXw)

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Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 4 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 6 We define $c_2Ebool_2E_27E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F$

Let $ty_2EternaryComparisons_2Eordering : \iota$ be given. Assume the following.

$$nonempty\ ty_2EternaryComparisons_2Eordering \quad (1)$$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (2)$$

Let $c_2EternaryComparisons_2Epair_compare : \iota \Rightarrow \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c.nonempty\ A_27c \Rightarrow \forall A_27d.nonempty\ A_27d \Rightarrow A_27a\ A_27b\ A_27c\ A_27d \in (((ty_2EternaryComparisons_2Eordering)^{(ty_2Epair_2Eprod\ A_27b\ A_27d)})^{(ty_2Epair_2Eprod\ A_27a\ A_27c)}) \quad (3)$$

Definition 7 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \quad (4)$$

Definition 8 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0x \in A_27a. \lambda V1y \in A_27b. (ap (c_2E$

Let $c_2EternaryComparisons_2EGREATER : \iota$ be given. Assume the following.

$$c_2EternaryComparisons_2EGREATER \in ty_2EternaryComparisons_2Eordering \quad (5)$$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2ESND\ A_27a\ A_27b \in (A_27b^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (6)$$

Let $c_2EternaryComparisons_2ELESS : \iota$ be given. Assume the following.

$$c_2EternaryComparisons_2ELESS \in ty_2EternaryComparisons_2Eordering \quad (7)$$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EFST\ A_27a\ A_27b \in (A_27a^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (8)$$

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \quad (9)$$

Let $c_2EternaryComparisons_2Eordering2num : \iota$ be given. Assume the following.

$$c_2EternaryComparisons_2Eordering2num \in (ty_2Enum_2Enum^{ty_2EternaryComparisons_2Eordering}) \quad (10)$$

Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in omega \quad (11)$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{omega}) \quad (12)$$

Definition 9 We define c_2Enum_2E0 to be $(ap\ c_2Enum_2EABS_num\ c_2Enum_2EZERO_REP)$.

Definition 10 We define $c_2Earithmetic_2EZERO$ to be c_2Enum_2E0 .

Let $c_2Enum_2EREP_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (omega^{ty_2Enum_2Enum}) \quad (13)$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (omega^{omega}) \quad (14)$$

Definition 11 We define c_2Enum_2ESUC to be $\lambda V0m \in ty_2Enum_2Enum.(ap\ c_2Enum_2EABS_num$
 Let $c_2Earithmetic_2E_2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2B \in ((ty_2Enum_2Enum)^{ty_2Enum_2Enum})^{ty_2Enum_2Enum} \quad (15)$$

Definition 12 We define $c_2Earithmetic_2EBIT1$ to be $\lambda V0n \in ty_2Enum_2Enum.(ap\ (ap\ c_2Earithmetic$

Definition 13 We define $c_2Earithmetic_2ENUMERAL$ to be $\lambda V0x \in ty_2Enum_2Enum.V0x$.

Definition 14 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p\ (ap\ P\ x)) \text{ then } (the\ (\lambda x.x \in A \wedge$
 of type $\iota \Rightarrow \iota$.

Definition 15 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.($

Definition 16 We define $c_2Ebool_2E_3F$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ V0P\ (ap\ (c_2Emin_2E_40$

Definition 17 We define $c_2Eprim_rec_2E_3C$ to be $\lambda V0m \in ty_2Enum_2Enum.\lambda V1n \in ty_2Enum_2Enum$

Definition 18 We define $c_2EternaryComparisons_2Eordering_CASE$ to be $\lambda A_27a : \iota.\lambda V0x \in ty_2Eternary$

Definition 19 We define $c_2Etoto_2EStrongLinearOrder_of_TO$ to be $\lambda A_27a : \iota.\lambda V0c \in ((ty_2EternaryC$

Definition 20 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in$

Definition 21 We define $c_2Epair_2EUNCURRY$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0f \in ((A_27c^{A_27$

Definition 22 We define c_2Epair_2ELEX to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0R1 \in ((2^{A_27a})^{A_27a}).\lambda V1R2 \in (($

Let $c_2EternaryComparisons_2EEQUAL : \iota$ be given. Assume the following.

$$c_2EternaryComparisons_2EEQUAL \in ty_2EternaryComparisons_2Eordering \quad (16)$$

Definition 23 We define $c_2Etoto_2ETO_of_LinearOrder$ to be $\lambda A_27a : \iota.\lambda V0r \in ((2^{A_27a})^{A_27a}).\lambda V1x \in$

Definition 24 We define $c_2Etoto_2ElexTO$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0R \in ((ty_2EternaryComparis$

Definition 25 We define $c_2Etoto_2ETotOrd$ to be $\lambda A_27a : \iota.\lambda V0c \in ((ty_2EternaryComparisons_2Eord$

Assume the following.

$$True \quad (17)$$

Assume the following.

$$\forall A_27a.\text{nonempty } A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (18)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\
& (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\
& (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (19)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\
& True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((\\
& (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (20)
\end{aligned}$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.((V0x = V0x) \Leftrightarrow True)) \quad (21)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (22)$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty \ A.27a \Rightarrow \forall A.27b.nonempty \ A.27b \Rightarrow (\\
& \forall V0f \in (A.27b^{A.27a}).(\forall V1g \in (A.27b^{A.27a}).((V0f = \\
& V1g) \Leftrightarrow (\forall V2x \in A.27a.((ap \ V0f \ V2x) = (ap \ V1g \ V2x)))))) \quad (23)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(\\
& p \ V0t)))))) \quad (24)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p \ V0t1) \Rightarrow \\
& ((p \ V1t2) \Rightarrow (p \ V2t3))) \Leftrightarrow (((p \ V0t1) \wedge (p \ V1t2)) \Rightarrow (p \ V2t3)))))) \quad (25)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in 2.(\forall V1x.27 \in 2.(\forall V2y \in 2.(\forall V3y.27 \in \\
& 2.(((p \ V0x) \Leftrightarrow (p \ V1x.27)) \wedge ((p \ V1x.27) \Rightarrow ((p \ V2y) \Leftrightarrow (p \ V3y.27)))))) \Rightarrow \\
& (((p \ V0x) \Rightarrow (p \ V2y)) \Leftrightarrow ((p \ V1x.27) \Rightarrow (p \ V3y.27)))))) \quad (26)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow \forall A.27c. \\
& \quad nonempty\ A.27c \Rightarrow \forall A.27d.nonempty\ A.27d \Rightarrow (\forall V0c1 \in (\\
& \quad (ty_2EternaryComparisons_2Eordering^{A.27b})^{A.27a}).(\forall V1c2 \in (\\
& \quad ((ty_2EternaryComparisons_2Eordering^{A.27d})^{A.27c}).(\forall V2x \in (\\
& \quad (ty_2Epair_2Eprod\ A.27a\ A.27c).(\forall V3y \in (ty_2Epair_2Eprod \\
& \quad A.27b\ A.27d).((ap\ (ap\ (ap\ (ap\ (c_2EternaryComparisons_2Epair_compare \\
& \quad A.27a\ A.27b\ A.27c\ A.27d)\ V0c1)\ V1c2)\ V2x)\ V3y) = (ap\ (ap\ (ap\ (ap\ (c_2EternaryComparisons_2Eordering \\
& \quad ty_2EternaryComparisons_2Eordering)\ (ap\ (ap\ V0c1\ (ap\ (c_2Epair_2EFST \\
& \quad A.27a\ A.27c)\ V2x))\ (ap\ (c_2Epair_2EFST\ A.27b\ A.27d)\ V3y)))\ c_2EternaryComparisons_2ELE \\
& \quad (ap\ (ap\ V1c2\ (ap\ (c_2Epair_2ESND\ A.27a\ A.27c)\ V2x))\ (ap\ (c_2Epair_2ESND \\
& \quad A.27b\ A.27d)\ V3y))))\ c_2EternaryComparisons_2EGREATER)))))) \\
& \hspace{15em} (27)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow (\\
& \quad \forall V0x \in A.27a.(\forall V1y \in A.27b.((ap\ (c_2Epair_2EFST\ A.27a \\
& \quad A.27b)\ (ap\ (ap\ (c_2Epair_2E_2C\ A.27a\ A.27b)\ V0x)\ V1y)) = V0x))) \\
& \hspace{15em} (28)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow (\\
& \quad \forall V0x \in A.27a.(\forall V1y \in A.27b.((ap\ (c_2Epair_2ESND\ A.27a \\
& \quad A.27b)\ (ap\ (ap\ (c_2Epair_2E_2C\ A.27a\ A.27b)\ V0x)\ V1y)) = V1y))) \\
& \hspace{15em} (29)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow (\\
& \quad \forall V0P \in (2^{(ty_2Epair_2Eprod\ A.27a\ A.27b)}).((\forall V1p \in \\
& \quad (ty_2Epair_2Eprod\ A.27a\ A.27b).(p\ (ap\ V0P\ V1p))) \Leftrightarrow (\forall V2p_1 \in \\
& \quad A.27a.(\forall V3p_2 \in A.27b.(p\ (ap\ V0P\ (ap\ (ap\ (c_2Epair_2E_2C \\
& \quad A.27a\ A.27b)\ V2p_1)\ V3p_2)))))) \\
& \hspace{15em} (30)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow (\\
& \quad \forall V0R \in ((ty_2EternaryComparisons_2Eordering^{A.27a})^{A.27a}). \\
& \quad (\forall V1V \in ((ty_2EternaryComparisons_2Eordering^{A.27b})^{A.27b}). \\
& \quad (((p\ (ap\ (c_2Etoto_2ETotOrd\ A.27a)\ V0R)) \wedge (p\ (ap\ (c_2Etoto_2ETotOrd \\
& \quad A.27b)\ V1V))) \Rightarrow (\forall V2x \in (ty_2Epair_2Eprod\ A.27a\ A.27b).(\\
& \quad \forall V3y \in (ty_2Epair_2Eprod\ A.27a\ A.27b).((ap\ (ap\ (ap\ (ap\ (c_2Etoto_2ElexTO \\
& \quad A.27a\ A.27b)\ V0R)\ V1V)\ V2x)\ V3y) = (ap\ (ap\ (ap\ (ap\ (c_2EternaryComparisons_2Eordering_CASE \\
& \quad ty_2EternaryComparisons_2Eordering)\ (ap\ (ap\ V0R\ (ap\ (c_2Epair_2EFST \\
& \quad A.27a\ A.27b)\ V2x))\ (ap\ (c_2Epair_2EFST\ A.27a\ A.27b)\ V3y)))\ c_2EternaryComparisons_2ELE \\
& \quad (ap\ (ap\ V1V\ (ap\ (c_2Epair_2ESND\ A.27a\ A.27b)\ V2x))\ (ap\ (c_2Epair_2ESND \\
& \quad A.27a\ A.27b)\ V3y))))\ c_2EternaryComparisons_2EGREATER)))))) \\
& \hspace{15em} (31)
\end{aligned}$$

Theorem 1

$$\begin{aligned} & \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow (\\ & \quad \forall V0R \in ((ty_2EternaryComparisons_2Eordering^{A_{.27a}})^{A_{.27a}}). \\ & \quad (\forall V1V \in ((ty_2EternaryComparisons_2Eordering^{A_{.27b}})^{A_{.27b}}). \\ & \quad ((p\ (ap\ (c_2Etoto_2ETotOrd\ A_{.27a})\ V0R)) \wedge (p\ (ap\ (c_2Etoto_2ETotOrd \\ & \quad A_{.27b})\ V1V))) \Rightarrow ((ap\ (ap\ (c_2EternaryComparisons_2Epair_compare \\ & \quad A_{.27a}\ A_{.27a}\ A_{.27b}\ A_{.27b})\ V0R)\ V1V) = (ap\ (ap\ (c_2Etoto_2ElexTO\ A_{.27a} \\ & \quad A_{.27b})\ V0R)\ V1V)))) \end{aligned}$$