

thm_2Econtainer_2EFINITE__BAG__OF__FMAP (TMdN3GT1NiRtKuMfDy6NxSchub2NzVewaNa)

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Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in \omega \tag{1}$$

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \tag{2}$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{\omega}) \tag{3}$$

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define c_2Enum_2E0 to be $(ap\ c_2Enum_2EABS_num\ c_2Enum_2EZERO_REP)$.

Definition 3 We define $c_2Earithmetic_2EZERO$ to be c_2Enum_2E0 .

Let $c_2Enum_2EREP_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (\omega^{ty_2Enum_2Enum}) \tag{4}$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (\omega^{\omega}) \tag{5}$$

Definition 4 We define c_2Ebool_2ET to be $(ap\ (ap\ (c_2Emin_2E_3D\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1x))$

Definition 5 We define $c_2Ebool_2E_21$ to be $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap\ (ap\ (c_2Emin_2E_3D\ (2^{A-27a}))\ (\lambda V1x \in 2.V1x))\ (\lambda V2x \in 2.V2x))$

Definition 6 We define c_2Enum_2ESUC to be $\lambda V0m \in ty_2Enum_2Enum.(ap\ c_2Enum_2EABS_num\ (ap\ (ap\ (c_2Emin_2E_3D\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1x))$

Let $c_2Earithmetic_2E_2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2B \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \tag{6}$$

Definition 7 We define $c_2Earithmic_2EBIT1$ to be $\lambda V0n \in ty_2Enum_2Enum.(ap (ap c_2Earithmic_2EBIT1))$.

Definition 8 We define $c_2Earithmic_2ENUMERAL$ to be $\lambda V0x \in ty_2Enum_2Enum.V0x$.

Definition 9 We define c_2Ebool_2EF to be $(ap (c_2Ebool_2E21 2)) (\lambda V0t \in 2.V0t)$.

Definition 10 We define $c_2Emin_2E3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 11 We define $c_2Ebool_2E2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E21 2)) (\lambda V2t \in 2.V0t2)))$.

Definition 12 We define c_2Emin_2E40 to be $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p (ap P x)) \text{ then } (the (\lambda x.x \in A \wedge P x))$ of type $\iota \Rightarrow \iota$.

Definition 13 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.V0t2)))$.

Definition 14 We define $c_2Ebag_2EBAG_INSERT$ to be $\lambda A_27a : \iota.\lambda V0e \in A_27a.\lambda V1b \in (ty_2Enum_2Enum A_27a)$.

Definition 15 We define $c_2Ecombin_2EK$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(\lambda V0x \in A_27a.(\lambda V1y \in A_27b.V0x))$.

Definition 16 We define $c_2Ebag_2EEMPTY_BAG$ to be $\lambda A_27a : \iota.(ap (c_2Ecombin_2EK ty_2Enum_2Enum A_27a))$.

Definition 17 We define $c_2Ebag_2EFINITE_BAG$ to be $\lambda A_27a : \iota.\lambda V0b \in (ty_2Enum_2Enum^{A_27a}).(ap (c_2Ecombin_2EK ty_2Enum_2Enum A_27a))$.

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \tag{7}$$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \tag{8}$$

Let $ty_2Efinite_map_2E fmap : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Efinite_map_2E fmap\ A0\ A1) \tag{9}$$

Let $c_2Efinite_map_2E fmap_REP : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efinite_map_2E fmap_REP\ A_27a\ A_27b \in (((ty_2Esum_2Esum\ A_27b\ ty_2Eone_2Eone)^{A_27a})(ty_2Efinite_map_2E fmap\ A_27a\ A_27b)) \tag{10}$$

Let $c_2Esum_2EOUTL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EOUTL\ A_27a\ A_27b \in (A_27a^{(ty_2Esum_2Esum\ A_27a\ A_27b)}) \tag{11}$$

Definition 18 We define $c_2Efinite_map_2EFAPPLY$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (ty_2Efinite_map_2E fmap A_27a A_27b)$.

Let $c_2Esum_2EISL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EISL \\ A_27a\ A_27b \in (2^{(ty_2Esum_2Esum\ A_27a\ A_27b)}) \end{aligned} \quad (12)$$

Definition 19 We define $c_2Efinite_map_2EFDOM$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (ty_2Efinite_map$

Definition 20 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap\ V1f\ V0x))$

Let $c_2Epred_set_2ECARD : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Epred_set_2ECARD\ A_27a \in (ty_2Enum_2Enum^{(2^{A_27a})}) \quad (13)$$

Definition 21 We define $c_2Econtainer_2EBAG_OF_FMAP$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0f \in$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (14)$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \quad (15)$$

Definition 22 We define c_2Epair_2E2C to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap\ (c_2E$

Let $c_2Efinite_map_2EFUPDATE : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efinite_map_2EFUPDATE\ A_27a\ A_27b \in (((ty_2Efinite_map_2E fmap\ A_27a\ A_27b)^{(ty_2Epair_2Eprod\ A_27a\ A_27b)})^{(ty_2Efinite_map$$

Definition 23 We define c_2Eone_2Eone to be $(ap\ (c_2Emin_2E_40\ ty_2Eone_2Eone))\ (\lambda V0x \in ty_2Eone_2E$

Definition 24 We define c_2Ebool_2E7E to be $(\lambda V0t \in 2.(ap\ (ap\ c_2Emin_2E_3D_3D_3E\ V0t))\ c_2Ebool_2E$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \quad (17)$$

Definition 25 We define c_2Esum_2EINR to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27b.(ap\ (c_2Esum_2EABS$

Let $c_2Efinite_map_2E fmap_ABS : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efinite_map_2E fmap_ABS\ A_27a\ A_27b \in ((ty_2Efinite_map_2E fmap\ A_27a\ A_27b)^{(ty_2Esum_2Esum\ A_27b\ ty_2Eone_2Eone)^{A_27a}}) \quad (18)$$

Definition 26 We define `c_2Efinite_map_2EFEMPTY` to be $\lambda A_27a : \iota. \lambda A_27b : \iota. (ap (c_2Efinite_map_2E$

Definition 27 We define `c_2Epred_set_2EEMPTY` to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a. c_2Ebool_2EF)$.

Definition 28 We define `c_2Ebool_2E_5C_2F` to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c_2Ebool_2E_21) 2) (\lambda V2t \in$

Let `c_2Epred_set_2EGSPEC` : $\iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a. nonempty\ A_27a \Rightarrow \forall A_27b. nonempty\ A_27b \Rightarrow c_2Epred_set_2EGSPEC \\ A_27a\ A_27b \in ((2^{A_27a})^{(ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b}}) \end{aligned} \quad (19)$$

Definition 29 We define `c_2Epred_set_2EINSERT` to be $\lambda A_27a : \iota. \lambda V0x \in A_27a. \lambda V1s \in (2^{A_27a}). (ap (c_2E$

Definition 30 We define `c_2Epred_set_2EUNIV` to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a. c_2Ebool_2EET)$.

Definition 31 We define `c_2Epred_set_2EDIFF` to be $\lambda A_27a : \iota. \lambda V0s \in (2^{A_27a}). \lambda V1t \in (2^{A_27a}). (ap (c_2E$

Definition 32 We define `c_2Epred_set_2ECOMPL` to be $\lambda A_27a : \iota. \lambda V0P \in (2^{A_27a}). (ap (ap (c_2Epred_set$

Let `c_2Efinite_map_2EDRESTRICT` : $\iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a. nonempty\ A_27a \Rightarrow \forall A_27b. nonempty\ A_27b \Rightarrow c_2Efinite_map_2EDRESTRICT \\ A_27a\ A_27b \in (((ty_2Efinite_map_2E fmap\ A_27a\ A_27b)^{(2^{A_27a})})^{(ty_2Efinite_map_2E fmap\ A_27a\ A_27b)}) \end{aligned} \quad (20)$$

Definition 33 We define `c_2Efinite_map_2Efdomsub` to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0fm \in (ty_2Efinite_map$

Assume the following.

$$\begin{aligned} \forall A_27a. nonempty\ A_27a \Rightarrow (p (ap (c_2Ebag_2EFINITE_BAG\ A_27a) \\ (c_2Ebag_2EEMPTY_BAG\ A_27a))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} \forall A_27a. nonempty\ A_27a \Rightarrow (\forall V0b \in (ty_2Enum_2Enum^{A_27a}). \\ ((p (ap (c_2Ebag_2EFINITE_BAG\ A_27a) V0b)) \Rightarrow (\forall V1e \in A_27a. \\ (p (ap (c_2Ebag_2EFINITE_BAG\ A_27a) (ap (ap (c_2Ebag_2EBAG_INSERT \\ A_27a) V1e) V0b)))))) \end{aligned} \quad (22)$$

Assume the following.

$$True \quad (23)$$

Assume the following.

$$\begin{aligned} (\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p \\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \end{aligned} \quad (24)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p\ V0t))) \quad (25)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (26)$$

Assume the following.

$$(\forall V0t \in 2. (((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \quad (27)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg(p\ V0t)))))) \quad (28)$$

Assume the following.

$$((\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True)))) \quad (29)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (30)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg(p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p\ V0t)))))) \quad (31)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p\ V0t1) \Rightarrow ((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \quad (32)$$

Assume the following.

$$(\forall V0x \in 2. (\forall V1x_27 \in 2. (\forall V2y \in 2. (\forall V3y_27 \in 2. (((p\ V0x) \Leftrightarrow (p\ V1x_27)) \wedge ((p\ V1x_27) \Rightarrow ((p\ V2y) \Leftrightarrow (p\ V3y_27)))) \Rightarrow (((p\ V0x) \Rightarrow (p\ V2y)) \Leftrightarrow ((p\ V1x_27) \Rightarrow (p\ V3y_27)))))) \quad (33)$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c. \\
& \quad nonempty\ A_27c \Rightarrow \forall A_27d.nonempty\ A_27d \Rightarrow \forall A_27e.nonempty \\
& \quad A_27e \Rightarrow \forall A_27f.nonempty\ A_27f \Rightarrow ((\forall V0f \in ((A_27c^{A_27b})^{A_27a}). \\
& \quad ((ap\ (ap\ (c_2Econtainer_2EBAG_OF_FMAP\ A_27c\ A_27a\ A_27b)\ V0f) \\
& \quad (c_2Efinite_map_2EFEMPTY\ A_27a\ A_27b)) = (c_2Ebag_2EEMPTY_BAG \\
& \quad A_27c))) \wedge (\forall V1f \in ((A_27f^{A_27e})^{A_27d}). (\forall V2b \in (ty_2Efinite_map_2E fmap \\
& \quad A_27d\ A_27e). (\forall V3k \in A_27d. (\forall V4v \in A_27e. ((ap\ (ap \\
& \quad (c_2Econtainer_2EBAG_OF_FMAP\ A_27f\ A_27d\ A_27e)\ V1f) (ap\ (ap \\
& \quad (c_2Efinite_map_2EFUPDATE\ A_27d\ A_27e)\ V2b) (ap\ (ap\ (c_2Epair_2E_2C \\
& \quad A_27d\ A_27e)\ V3k)\ V4v))) = (ap\ (ap\ (c_2Ebag_2EBAG_INSERT\ A_27f) \\
& \quad (ap\ (ap\ V1f\ V3k)\ V4v)) (ap\ (ap\ (c_2Econtainer_2EBAG_OF_FMAP\ A_27f \\
& \quad A_27d\ A_27e)\ V1f) (ap\ (ap\ (c_2Efinite_map_2Efdomsb\ A_27d\ A_27e) \\
& \quad V2b)\ V3k))))))))))
\end{aligned} \tag{34}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0P \in (2^{(ty_2Efinite_map_2E fmap\ A_27a\ A_27b)}). ((p \\
& \quad (ap\ V0P\ (c_2Efinite_map_2EFEMPTY\ A_27a\ A_27b))) \wedge (\forall V1f \in \\
& \quad (ty_2Efinite_map_2E fmap\ A_27a\ A_27b). ((p\ (ap\ V0P\ V1f)) \Rightarrow (\forall V2x \in \\
& \quad A_27a. (\forall V3y \in A_27b. ((\neg(p\ (ap\ (ap\ (c_2Ebool_2EIN\ A_27a) \\
& \quad V2x) (ap\ (c_2Efinite_map_2EFDOM\ A_27a\ A_27b)\ V1f)))) \Rightarrow (p\ (ap\ V0P \\
& \quad (ap\ (ap\ (c_2Efinite_map_2EFUPDATE\ A_27a\ A_27b)\ V1f) (ap\ (ap\ (c_2Epair_2E_2C \\
& \quad A_27a\ A_27b)\ V2x)\ V3y)))))))))) \Rightarrow (\forall V4f \in (ty_2Efinite_map_2E fmap \\
& \quad A_27a\ A_27b). (p\ (ap\ V0P\ V4f))))))
\end{aligned} \tag{35}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0k \in A_27a. (\forall V1fm \in (ty_2Efinite_map_2E fmap\ A_27a \\
& \quad A_27b). ((\neg(p\ (ap\ (ap\ (c_2Ebool_2EIN\ A_27a)\ V0k) (ap\ (c_2Efinite_map_2EFDOM \\
& \quad A_27a\ A_27b)\ V1fm)))) \Rightarrow ((ap\ (ap\ (c_2Efinite_map_2Efdomsb\ A_27a \\
& \quad A_27b)\ V1fm)\ V0k) = V1fm))))))
\end{aligned} \tag{36}$$

Theorem 1

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c. \\
& \quad nonempty\ A_27c \Rightarrow (\forall V0f \in ((A_27c^{A_27b})^{A_27a}). (\forall V1b \in \\
& \quad (ty_2Efinite_map_2E fmap\ A_27a\ A_27b). (p\ (ap\ (c_2Ebag_2EFINITE_BAG \\
& \quad A_27c) (ap\ (ap\ (c_2Econtainer_2EBAG_OF_FMAP\ A_27c\ A_27a\ A_27b) \\
& \quad V0f)\ V1b))))))
\end{aligned}$$