thm_2Eenumeral_2EOU__ASSOC (TMLC6yZpXBQQsd16DSahmCTFrdkBvZwW1C6)

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Let $ty_2EternaryComparisons_2Eordering : \iota$ be given. Assume the following.

$$nonempty\ ty_2EternaryComparisons_2Eordering \tag{1}$$

Let $c_2EternaryComparisons_2ELESS : \iota$ be given. Assume the following.

$$c_2EternaryComparisons_2ELESS \in ty_2EternaryComparisons_2Eordering \tag{2}$$

Let $ty_2Etoto_2Etoto : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Etoto_2Etoto\ A0) \tag{3}$$

Let $c_2Etoto_2Eapto : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Etoto_2Eapto\ A_27a \in (((ty_2EternaryComparisons_2Eordering^{A_27a})^{A_2} \tag{4}$$

Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj__o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2EIN *to be* $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap\ V1f\ V0x)))$

Definition 3 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj__o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 4 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c_2Emin_2E_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 5 *We define* c_2Ebool_2E_21 *to be* $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ (ap\ (\mathsf{c_2Emin_2E_3D}\ (2^{A_27}$

Definition 6 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c_2Ebool_2E_21}\ 2)\ (\lambda V2t \in 2$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \tag{5}$$

Let $c_2Epair_2EABS__prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS__prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})}) \tag{6}$$

Definition 7 *We define* c_2Epair_2E_2C *to be* $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap\ (c_2E$

Let $c_2Epred__set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epred__set_2EGSPEC\ A_27a\ A_27b \in ((2^{A_27a})^{((ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b})}) \tag{7}$$

Definition 8 *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c_2Ebool_2E_21}\ 2)\ (\lambda V2t \in 2$

Definition 9 *We define* c_2Epred__set_2EUNION *to be* $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap\ (c_2$

Definition 10 *We define* c_2Eenumeral_2EOU *to be* $\lambda A_27a : \iota.\lambda V0cmp \in (ty_2Etoto_2Etoto\ A_27a).\lambda V1t$

Definition 11 *We define* c_2Eenumeral_2ELESS__ALL *to be* $\lambda A_27a : \iota.\lambda V0cmp \in (ty_2Etoto_2Etoto\ A_27a$

Definition 12 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 13 *We define* c_2Ebool_2E_3F *to be* $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ V0P\ (ap\ (\mathsf{c_2Emin_2E_40}$

Definition 14 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c_2Ebool_2E_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 15 *We define* c_2Ebool_2ECOND *to be* $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.($

Let $c_2EternaryComparisons_2EGREATER : \iota$ be given. Assume the following.

$$c_2EternaryComparisons_2EGREATER \in ty_2EternaryComparisons_2Eordering \tag{8}$$

Definition 16 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c_2Emin_2E_3D_3D_3E}\ V0t)\ \mathsf{c_2Ebool_2E}$

Let $c_2EternaryComparisons_2EEQUAL : \iota$ be given. Assume the following.

$$c_2EternaryComparisons_2EEQUAL \in ty_2EternaryComparisons_2Eordering \tag{9}$$

Assume the following.

$$True \tag{10}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{11}$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \tag{12}$$

Assume the following.

$$(\forall V0t \in 2.((p\ V0t) \vee (\neg (p\ V0t)))) \tag{13}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \tag{14}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge ((((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t))))))) \tag{15}$$

Assume the following.

$$(\forall V0t \in 2.(((True \vee (p\ V0t)) \Leftrightarrow True) \wedge ((((p\ V0t) \vee True) \Leftrightarrow True) \wedge (((False \vee (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \vee False) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee (p\ V0t)) \Leftrightarrow (p\ V0t))))))) \tag{16}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))))) \tag{17}$$

Assume the following.

$$((\forall V0t \in 2.((\neg (\neg (p\ V0t))) \Leftrightarrow (p\ V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \tag{18}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a.((V0x = V0x) \Leftrightarrow True)) \tag{19}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a.(\forall V1y \in A_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{20}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg (p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (p\ V0t))))))) \tag{21}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t1 \in A_27a.(\forall V1t2 \in A_27a.(((ap\ (ap\ (ap\ (\mathsf{c_2Ebool_2ECOND}\ A_27a)\ \mathsf{c_2Ebool_2ET})\ V0t1)\ V1t2) = V0t1) \wedge ((ap\ (ap\ (ap\ (\mathsf{c_2Ebool_2ECOND}\ A_27a)\ \mathsf{c_2Ebool_2EF})\ V0t1)\ V1t2) = V1t2)))) \tag{22}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p\ V0A) \vee (p\ V1B) \wedge (p\ V2C))) \Leftrightarrow (((p\ V0A) \vee (p\ V1B)) \wedge ((p\ V0A) \vee (p\ V2C))))))) \tag{23}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.((((p\ V1B) \wedge (p\ V2C)) \vee (p\ V0A)) \Leftrightarrow (((p\ V1B) \vee (p\ V0A)) \wedge ((p\ V2C) \vee (p\ V0A))))))) \tag{24}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p\ V0t1) \Rightarrow ((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \tag{25}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in 2.(\forall V2x \in A_27a.(\forall V3x_27 \in A_27a.(\forall V4y \in A_27a.(\forall V5y_27 \in A_27a.((((p\ V0P) \Leftrightarrow (p\ V1Q)) \wedge (((p\ V1Q) \Rightarrow (V2x = V3x_27)) \wedge ((\neg(p\ V1Q)) \Rightarrow (V4y = V5y_27)))) \Rightarrow ((ap\ (ap\ (ap\ (\mathsf{c_2Ebool_2ECOND}\ A_27a)\ V0P)\ V2x)\ V4y) = (ap\ (ap\ (ap\ (\mathsf{c_2Ebool_2ECOND}\ A_27a)\ V1Q)\ V3x_27)\ V5y_27))))))))) \tag{26}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0cmp \in (ty_2Etoto_2Etoto\ A_27a).(\forall V1x \in A_27a.(\forall V2u \in (2^{A_27a}).(\forall V3v \in (2^{A_27a}).((p\ (ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2ELESS__ALL}\ A_27a)\ V0cmp)\ V1x)\ (ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2EOU}\ A_27a)\ V0cmp)\ V2u)\ V3v))) \Leftrightarrow (p\ (ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2ELESS__ALL}\ A_27a)\ V0cmp)\ V1x)\ V2u)) \wedge (p\ (ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2ELESS__ALL}\ A_27a)\ V0cmp)\ V1x)\ V3v)))))))) \tag{27}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\forall V0x \in A_27a.(\forall V1y \in A_27b.(\forall V2a \in A_27a.(\forall V3b \in A_27b.(((ap\ (ap\ (\mathsf{c_2Epair_2E_2C}\ A_27a\ A_27b)\ V0x)\ V1y) = (ap\ (ap\ (\mathsf{c_2Epair_2E_2C}\ A_27a\ A_27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b))))))) \tag{28}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0s \in (2^{A_27a}).(\forall V1t \in (2^{A_27a}).((V0s = V1t) \Leftrightarrow (\forall V2x \in A_27a.((p\ (ap\ (ap\ (\mathsf{c_2Ebool_2EIN}\ A_27a)\ V2x)\ V0s)) \Leftrightarrow (p\ (ap\ (ap\ (\mathsf{c_2Ebool_2EIN}\ A_27a)\ V2x)\ V1t))))))) \tag{29}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\forall V0f \in ((ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b}).(\forall V1v \in A_27a.((p\ (ap\ (ap\ (\mathsf{c_2Ebool_2EIN}\ A_27a)\ V1v)\ (ap\ (c_2Epred__set_2EGSPEC\ A_27a\ A_27b)\ V0f))) \Leftrightarrow (\exists V2x \in A_27b.((ap\ (ap\ (\mathsf{c_2Epair_2E_2C}\ A_27a\ 2)\ V1v)\ \mathsf{c_2Ebool_2ET}) = (ap\ V0f\ V2x)))))) \tag{30}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0s \in (2^{A_27a}).(\forall V1t \in (2^{A_27a}).(\forall V2x \in A_27a.((p\ (ap\ (ap\ (\mathsf{c_2Ebool_2EIN}\ A_27a)\ V2x)\ (ap\ (ap\ (\mathsf{c_2Epred__set_2EUNION}\ A_27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap\ (ap\ (\mathsf{c_2Ebool_2EIN}\ A_27a)\ V2x)\ V0s)) \vee (p\ (ap\ (ap\ (\mathsf{c_2Ebool_2EIN}\ A_27a)\ V2x)\ V1t))))))) \tag{31}$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \tag{32}$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \tag{33}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))) \tag{34}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))) \tag{35}$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \tag{36}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))) \wedge ((p\ V2r) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V0p)))))))))) \tag{37}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow ((p\ V1q) \wedge (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V2r)))) \wedge (((p\ V1q) \vee (\neg(p\ V0p))) \wedge ((p\ V2r) \vee (\neg(p\ V0p)))))))) \tag{38}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow ((p\ V1q) \vee (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee (\neg(p\ V1q))) \wedge (((p\ V0p) \vee (\neg(p\ V2r))) \wedge ((p\ V1q) \vee ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \tag{39}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow ((p\ V1q) \Rightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee (p\ V1q)) \wedge (((p\ V0p) \vee (\neg(p\ V2r))) \wedge ((\neg(p\ V1q)) \vee ((p\ V2r) \vee (\neg(p\ V0p)))))))))) \tag{40}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(((p\ V0p) \Leftrightarrow (\neg(p\ V1q))) \Leftrightarrow (((p\ V0p) \vee (p\ V1q)) \wedge ((\neg(p\ V1q)) \vee (\neg(p\ V0p))))))) \tag{41}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(\forall V3s \in 2.(((p\ V0p) \Leftrightarrow (p\ (ap\ (ap\ (ap\ (\mathsf{c_2Ebool_2ECOND}\ 2)\ V1q)\ V2r)\ V3s))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (\neg(p\ V3s)))) \wedge (((p\ V0p) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V1q)))) \wedge (((p\ V0p) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V3s)))) \wedge (((\neg(p\ V1q)) \vee ((p\ V2r) \vee (\neg(p\ V0p)))) \wedge ((p\ V1q) \vee ((p\ V3s) \vee (\neg(p\ V0p)))))))))))) \tag{42}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p) \Rightarrow (p\ V1q))) \Rightarrow (p\ V0p)))) \tag{43}$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p\ V0p) \Rightarrow (p\ V1q))) \Rightarrow (\neg(p\ V1q))))) \tag{44}$$

Assume the following.

$$(((\neg(c_2EternaryComparisons_2ELESS = c_2EternaryComparisons_2EEQUAL)) \wedge ((\neg(c_2EternaryComparisons_2ELESS = c_2EternaryComparisons_2EGREATER)) \wedge (\neg(c_2EternaryComparisons_2EEQUAL = c_2EternaryComparisons_2EGREATER)))) \wedge ((\neg(c_2EternaryComparisons_2EEQUAL = c_2EternaryComparisons_2ELESS)) \wedge ((\neg(c_2EternaryComparisons_2EGREATER = c_2EternaryComparisons_2ELESS)) \wedge (\neg(c_2EternaryComparisons_2EGREATER = c_2EternaryComparisons_2EEQUAL))))) \tag{45}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0c \in (ty_2Etoto_2Etoto\ A_27a).(\forall V1x \in A_27a.((ap\ (ap\ (ap\ (c_2Etoto_2Eapto\ A_27a)\ V0c)\ V1x)\ V1x) = c_2EternaryComparisons_2EEQUAL))) \tag{46}$$

Theorem 1

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0cmp \in (ty_2Etoto_2Etoto\ A_27a).(\forall V1a \in (2^{A_27a}).(\forall V2b \in (2^{A_27a}).(\forall V3c \in (2^{A_27a}).((ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2EOU}\ A_27a)\ V0cmp)\ V1a)\ (ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2EOU}\ A_27a)\ V0cmp)\ V2b)\ V3c)) = (ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2EOU}\ A_27a)\ V0cmp)\ (ap\ (ap\ (ap\ (\mathsf{c_2Eenumeral_2EOU}\ A_27a)\ V0cmp)\ V1a)\ V2b))\ V3c))))))$$