

thm_2ErrorStateMonad_2EMMAP_ID
 (TMGEGKGsGeENYVEQJKDsVe-
 bkNE83687hmCY)

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Definition 1 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.$ if $(\exists x \in A.p (ap P x))$ then (the $(\lambda x.x \in A \wedge p$
 of type $\iota \Rightarrow \iota$).

Definition 2 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$
 of type $\iota \Rightarrow \iota$.

Definition 3 We define c_2Ebool_2ET to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 4 We define $c_2Ecombin_2EK$ to be $\lambda A.\lambda 27a : \iota.\lambda A.\lambda 27b : \iota.(\lambda V0x \in A.27a.(\lambda V1y \in A.27b.V0x))$

Definition 5 We define $c_2Ecombin_2ES$ to be $\lambda A.\lambda 27a : \iota.\lambda A.\lambda 27b : \iota.\lambda A.\lambda 27c : \iota.(\lambda V0f \in ((A.27c^{A.27b})^{A.27a}))$

Definition 6 We define $c_2Ecombin_2EI$ to be $\lambda A.\lambda 27a : \iota.(ap (ap (c_2Ecombin_2ES A.27a (A.27a^{A.27a})) A.27a))$

Definition 7 We define $c_2Emin_2E_3D.3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$
 of type ι .

Definition 8 We define $c_2Ebool_2E_21$ to be $\lambda A.\lambda 27a : \iota.(\lambda V0P \in (2^{A.27a}).(ap (ap (c_2Emin_2E_3D (2^{A.27a})) P) P))$

Definition 9 We define $c_2Ebool_2E_2F.5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t)))$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \tag{1}$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow \forall A.27b.nonempty A.27b \Rightarrow c_2Epair_2EABS_prod A.27a A.27b \in ((ty_2Epair_2Eprod A.27a A.27b)^{(2^{A.27b})^{A.27a}}) \tag{2}$$

Definition 10 We define $c_2Epair_2E_2C$ to be $\lambda A.\lambda 27a : \iota.\lambda A.\lambda 27b : \iota.\lambda V0x \in A.27a.\lambda V1y \in A.27b.(ap (c_2Epair_2E_2C$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \quad (3)$$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \quad (4)$$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \quad (5)$$

Definition 11 We define c_2Esum_2EINL to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27a.(ap\ (c_2Esum_2EABS_sum\ A_27a\ A_27b)\ V0e)$

Let $ty_2Eoption_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Eoption_2Eoption\ A0) \quad (6)$$

Let $c_2Eoption_2Eoption_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Eoption_2Eoption_ABS\ A_27a \in ((ty_2Eoption_2Eoption\ A_27a)^{(ty_2Esum_2Esum\ A_27a\ ty_2Eone_2Eone)}) \quad (7)$$

Definition 12 We define $c_2Eoption_2ESOME$ to be $\lambda A_27a : \iota.\lambda V0x \in A_27a.(ap\ (c_2Eoption_2Eoption_ABS\ A_27a)\ V0x)$

Definition 13 We define $c_2ErrorStateMonad_2EUNIT$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27b.(\lambda V1s \in A_27a.(ap\ (c_2Eoption_2ESOME\ A_27a)\ V1s)\ V0x)$

Definition 14 We define $c_2Ecombin_2Eo$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0f \in (A_27b^{A_27c}).\lambda V1g \in (A_27a^{A_27c}).(ap\ (c_2Eoption_2ESOME\ A_27a)\ V1g)\ V0f$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2ESND\ A_27a\ A_27b \in (A_27b^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (8)$$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EFST\ A_27a\ A_27b \in (A_27a^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (9)$$

Definition 15 We define $c_2Epair_2Epair_CASE$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0p \in (ty_2Epair_2ESND\ A_27a\ A_27b).\lambda V1q \in (ty_2Epair_2ESND\ A_27b\ A_27c).(ap\ (c_2Epair_2EFST\ A_27a\ A_27c)\ V1q)\ V0p$

Definition 16 We define c_2Eone_2Eone to be $(ap\ (c_2Emin_2E_40\ ty_2Eone_2Eone)\ (\lambda V0x \in ty_2Eone_2Eone.\lambda V1y \in ty_2Eone_2Eone.(ap\ (c_2Epair_2Epair_CASE\ A_27a\ A_27b)\ V1y)\ V0x))$

Definition 17 We define c_2Ebool_2EF to be $(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 18 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap\ (ap\ (c_2Emin_2E_3D_3D_3E\ V0t)\ c_2Ebool_2EF)\ V0t))$

Definition 19 We define c_2Esum_2EINR to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27b.(ap (c_2Esum_2EABS$

Definition 20 We define $c_2Eoption_2ENONE$ to be $\lambda A_27a : \iota.(ap (c_2Eoption_2Eoption_ABS A_27a) (c$

Let $c_2Eoption_2Eoption_CASE : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Eoption_2Eoption_CASE \\ A_27a A_27b \in (((A_27b^{(A_27b^{A_27a})})^{A_27b})^{(ty_2Eoption_2Eoption A_27a)}) \end{aligned} \quad (10)$$

Definition 21 We define $c_2ErrorStateMonad_2EBIND$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0g \in ((ty$

Definition 22 We define $c_2ErrorStateMonad_2EMMAP$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0f \in (A$

Definition 23 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in$

Definition 24 We define $c_2Ebool_2E_3F$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap V0P (ap (c_2Emin_2E_40$

Assume the following.

$$True \quad (11)$$

Assume the following.

$$\begin{aligned} (\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p \\ V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \end{aligned} \quad (12)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (13)$$

Assume the following.

$$\begin{aligned} \forall A_27a.nonempty A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in \\ A_27a.(p V0t)) \Leftrightarrow (p V0t))) \end{aligned} \quad (14)$$

Assume the following.

$$\begin{aligned} \forall A_27a.nonempty A_27a \Rightarrow (\forall V0t \in 2.((\exists V1x \in \\ A_27a.(p V0t)) \Leftrightarrow (p V0t))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge \\ (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee \\ (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((\\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (19)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a.(V0x = V0x)) \quad (20)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a.((V0x = V0x) \Leftrightarrow True)) \quad (21)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a.(\forall V1y \in \\ & A_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty \ A_27a \Rightarrow \forall A_27b.nonempty \ A_27b \Rightarrow (\\ & \forall V0f \in (A_27b^{A_27a}).(\forall V1g \in (A_27b^{A_27a}).((V0f = \\ & V1g) \Leftrightarrow (\forall V2x \in A_27a.((ap \ V0f \ V2x) = (ap \ V1g \ V2x)))))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(\\ & p \ V0t)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((p \ V0A) \vee (p \ V1B)) \Leftrightarrow ((p \ V1B) \vee \\ & (p \ V0A)))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p \ V0t1) \Rightarrow \\ & (p \ V1t2) \Rightarrow (p \ V2t3))) \Leftrightarrow (((p \ V0t1) \wedge (p \ V1t2)) \Rightarrow (p \ V2t3)))))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0P \in 2.(\forall V1P_27 \in 2.(\forall V2Q \in 2.(\forall V3Q_27 \in \\ & 2.(((p \ V2Q) \Rightarrow ((p \ V0P) \Leftrightarrow (p \ V1P_27))) \wedge ((p \ V1P_27) \Rightarrow ((p \ V2Q) \Leftrightarrow (p \ V3Q_27)))) \Rightarrow \\ & (((p \ V0P) \wedge (p \ V2Q)) \Leftrightarrow ((p \ V1P_27) \wedge (p \ V3Q_27)))))) \end{aligned} \quad (27)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0P \in (2^{A_27a}). (\forall V1a \in A_27a. ((\exists V2x \in A_27a. ((V2x = V1a) \wedge (p (ap\ V0P\ V2x)))) \Leftrightarrow (p (ap\ V0P\ V1a)))))) \quad (28)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. ((ap (c_2Ecombin_2EI\ A_27a)\ V0x) = V0x)) \quad (29)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\forall V0f \in (A_27b^{A_27a}). (((ap (ap (c_2Ecombin_2Eo\ A_27a\ A_27b\ A_27b) (c_2Ecombin_2EI\ A_27b))\ V0f) = V0f) \wedge ((ap (ap (c_2Ecombin_2Eo\ A_27a\ A_27b\ A_27a)\ V0f) (c_2Ecombin_2EI\ A_27a)) = V0f)))) \quad (30)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0opt \in (ty_2Eoption_2Eoption\ A_27a). ((V0opt = (c_2Eoption_2ENONE\ A_27a)) \vee (\exists V1x \in A_27a. (V0opt = (ap (c_2Eoption_2ESOME\ A_27a)\ V1x)))))) \quad (31)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\forall V0v \in A_27b. (\forall V1f \in (A_27b^{A_27a}). ((ap (ap (ap (c_2Eoption_2Eoption_CASE\ A_27a\ A_27b) (c_2Eoption_2ENONE\ A_27a))\ V0v)\ V1f) = V0v))) \wedge (\forall V2x \in A_27a. (\forall V3v \in A_27b. (\forall V4f \in (A_27b^{A_27a}). ((ap (ap (ap (c_2Eoption_2Eoption_CASE\ A_27a\ A_27b) (ap (c_2Eoption_2ESOME\ A_27a)\ V2x))\ V3v)\ V4f) = (ap\ V4f\ V2x))))))) \quad (32)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. (((ap (c_2Eoption_2ESOME\ A_27a)\ V0x) = (ap (c_2Eoption_2ESOME\ A_27a)\ V1y)) \Leftrightarrow (V0x = V1y)))) \quad (33)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. (\neg((c_2Eoption_2ENONE\ A_27a) = (ap (c_2Eoption_2ESOME\ A_27a)\ V0x)))) \quad (34)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0opt \in (ty_2Eoption_2Eoption\ A_27a). ((\exists V1x \in A_27a. (V0opt = (ap (c_2Eoption_2ESOME\ A_27a)\ V1x))) \vee (V0opt = (c_2Eoption_2ENONE\ A_27a)))) \quad (35)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \quad \forall V0x \in A_27a. (\forall V1y \in A_27b. (\forall V2a \in A_27a. (\forall V3b \in \\ & \quad A_27b. (((ap\ (ap\ (c_2Epair_2E_2C\ A_27a\ A_27b)\ V0x)\ V1y) = (ap\ (ap \\ & \quad (c_2Epair_2E_2C\ A_27a\ A_27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b)))))) \\ & \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \quad \forall V0x \in (ty_2Epair_2Eprod\ A_27a\ A_27b). (\exists V1q \in A_27a. \\ & \quad (\exists V2r \in A_27b. (V0x = (ap\ (ap\ (c_2Epair_2E_2C\ A_27a\ A_27b)\ \\ & \quad V1q)\ V2r)))))) \\ & \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c. \\ & \quad nonempty\ A_27c \Rightarrow (\forall V0x \in A_27b. (\forall V1y \in A_27c. (\forall V2f \in \\ & \quad ((A_27a^{A_27c})^{A_27b}). ((ap\ (ap\ (c_2Epair_2Epair_CASE\ A_27a\ A_27b \\ & \quad A_27c)\ (ap\ (ap\ (c_2Epair_2E_2C\ A_27b\ A_27c)\ V0x)\ V1y))\ V2f) = (ap \\ & \quad (ap\ V2f\ V0x)\ V1y)))))) \\ & \end{aligned} \quad (38)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \quad (39)$$

Assume the following.

$$(\forall V0A \in 2. ((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \quad (40)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \quad (41)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p\ V0p) \Rightarrow (p\ V1q))) \Rightarrow (p\ V0p)))) \quad (42)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p\ V0p) \Rightarrow (p\ V1q))) \Rightarrow (\neg(p\ V1q)))))) \quad (43)$$

Theorem 1

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \quad (ap\ (c_2ErrorStateMonad_2EMMAP\ A_27a\ A_27b\ A_27b)\ (c_2Ecombin_2EI \\ & \quad A_27b)) = (c_2Ecombin_2EI\ ((ty_2Eoption_2Eoption\ (ty_2Epair_2Eprod \\ & \quad A_27b\ A_27a))^{A_27a}))) \\ & \end{aligned}$$