

# thm\_2Eextreal\_2EEXTREAL\_\_SUM\_\_IMAGE\_\_FINITE\_\_CONST (TMdnnEYhXZU1jU9CJKq25ab8qtvS49i1nGV)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})))$

**Definition 4** We define  $c\_2Ebool\_2E\_EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2E\_EF)$ .

**Definition 6** We define  $c\_2Ebool\_2E\_EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap V1f V0x)))$

**Definition 7** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 8** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t)))$

**Definition 9** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t)))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \tag{2}$$

**Definition 10** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V2z \in 2.V2z))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}}) \quad (3)$$

**Definition 11** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ s)\ V0x)$

**Definition 12** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap\ (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E\_7E\ t))$

**Definition 13** We define  $c\_2Epred\_set\_2EDIFF$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ s)\ t)$

**Definition 14** We define  $c\_2Epred\_set\_2EDELETE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1x \in A\_27a.(ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ s)\ x)$

Let  $ty\_2Eextreal\_2Eextreal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eextreal\_2Eextreal \quad (4)$$

Let  $c\_2Eextreal\_2Eextreal\_add : \iota$  be given. Assume the following.

$$c\_2Eextreal\_2Eextreal\_add \in ((ty\_2Eextreal\_2Eextreal^{ty\_2Eextreal\_2Eextreal})^{ty\_2Eextreal\_2Eextreal}) \quad (5)$$

Let  $ty\_2Erealax\_2Ereal : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Erealax\_2Ereal \quad (6)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (7)$$

Let  $c\_2Ereal\_2Ereal\_of\_num : \iota$  be given. Assume the following.

$$c\_2Ereal\_2Ereal\_of\_num \in (ty\_2Erealax\_2Ereal^{ty\_2Enum\_2Enum}) \quad (8)$$

Let  $c\_2Eextreal\_2ENormal : \iota$  be given. Assume the following.

$$c\_2Eextreal\_2ENormal \in (ty\_2Eextreal\_2Eextreal^{ty\_2Erealax\_2Ereal}) \quad (9)$$

**Definition 15** We define  $c\_2Eextreal\_2Eextreal\_of\_num$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ c\_2Eextreal\_2ENormal\ n)$

Let  $c\_2Eextreal\_2Eextreal\_mul : \iota$  be given. Assume the following.

$$c\_2Eextreal\_2Eextreal\_mul \in ((ty\_2Eextreal\_2Eextreal^{ty\_2Eextreal\_2Eextreal})^{ty\_2Eextreal\_2Eextreal}) \quad (10)$$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (11)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (12)$$

**Definition 16** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c\_2Epred\_set\_2EITSET : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EITSET \\ A\_27a\ A\_27b \in (((A\_27b^{A\_27b})^{(2^{A\_27a})})^{((A\_27b^{A\_27b})^{A\_27a})}) \end{aligned} \quad (13)$$

**Definition 17** We define  $c\_2Eextreal\_2EEXTREAL\_SUM\_IMAGE$  to be  $\lambda A\_27a : \iota.\lambda V0f \in (ty\_2Eextreal\_2EEXTREAL)$ .

**Definition 18** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).(ap\ (c\_2Ebool\_2E21\ 2))$ .

**Definition 19** We define  $c\_2Emin\_2E40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p\ (ap\ P\ x)) \text{ then } (the\ (\lambda x.x \in A \wedge P\ x)) \text{ of type } \iota \Rightarrow \iota$ .

**Definition 20** We define  $c\_2Ebool\_2E3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E40\ A\_27a\ P))))$ .

Let  $c\_2Epred\_set\_2ECARD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epred\_set\_2ECARD\ A\_27a \in (ty\_2Enum\_2Enum^{(2^{A\_27a})}) \quad (14)$$

Assume the following.

$$True \quad (15)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (16)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow \\ (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \\ (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \vee (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \vee True) \Leftrightarrow True) \wedge \\ (((False \vee (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee False) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee \\ (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow \\ True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge \\ ((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))) \end{aligned} \quad (19)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (20)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (21)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg(p\ V0t)))))) \quad (22)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p\ V0t1) \Rightarrow (p\ V1t2) \Rightarrow (p\ V2t3)) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \quad (23)$$

Assume the following.

$$(\forall V0x \in 2. (\forall V1x\_27 \in 2. (\forall V2y \in 2. (\forall V3y\_27 \in 2. (((p\ V0x) \Leftrightarrow (p\ V1x\_27)) \wedge ((p\ V1x\_27) \Rightarrow ((p\ V2y) \Leftrightarrow (p\ V3y\_27)))))) \Rightarrow (((p\ V0x) \Rightarrow (p\ V2y)) \Leftrightarrow ((p\ V1x\_27) \Rightarrow (p\ V3y\_27)))))) \quad (24)$$

Assume the following.

$$(\forall V0x \in ty\_2Eextreal\_2Eextreal. ((ap\ (ap\ c\_2Eextreal\_2Eextreal\_mul\ (ap\ c\_2Eextreal\_2Eextreal\_of\_num\ c\_2Enum\_2E0))\ V0x) = (ap\ c\_2Eextreal\_2Eextreal\_of\_num\ c\_2Enum\_2E0))) \quad (25)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0f \in (ty\_2Eextreal\_2Eextreal^{A\_27a}). (((ap\ (ap\ (c\_2Eextreal\_2EEXTREAL\_SUM\_IMAGE\ A\_27a)\ V0f)\ (c\_2Epred\_set\_2EEMPTY\ A\_27a)) = (ap\ c\_2Eextreal\_2Eextreal\_of\_num\ c\_2Enum\_2E0)) \wedge (\forall V1e \in A\_27a. (\forall V2s \in (2^{A\_27a}). ((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V2s)) \Rightarrow ((ap\ (ap\ (c\_2Eextreal\_2EEXTREAL\_SUM\_IMAGE\ A\_27a)\ V0f)\ (ap\ (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V1e)\ V2s)) = (ap\ (ap\ c\_2Eextreal\_2Eextreal\_add\ (ap\ V0f\ V1e))\ (ap\ (ap\ (c\_2Eextreal\_2EEXTREAL\_SUM\_IMAGE\ A\_27a)\ V0f)\ (ap\ (ap\ (c\_2Epred\_set\_2EDELETE\ A\_27a)\ V2s)\ V1e)))))))))) \quad (26)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). ((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V0s)) \Rightarrow (\forall V1f \in (ty\_2Eextreal\_2Eextreal^{A\_27a}). (\forall V2p \in A\_27a. (((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2p)\ V0s)) \wedge (\forall V3q \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V3q)\ V0s)) \Rightarrow ((ap\ V1f\ V2p) = (ap\ V1f\ V3q)))))) \Rightarrow ((ap\ (ap\ (c\_2Eextreal\_2EEXTREAL\_SUM\_IMAGE\ A\_27a)\ V1f)\ V0s) = (ap\ (ap\ c\_2Eextreal\_2Eextreal\_mul\ (ap\ c\_2Eextreal\_2Eextreal\_of\_num\ (ap\ (c\_2Epred\_set\_2ECARD\ A\_27a)\ V0s))\ (ap\ V1f\ V2p)))))) \quad (27)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in \\ A\_27a. (\forall V2s \in (2^{A-27a}). ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a) \\ V0x)\ (ap\ (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V1y)\ V2s)))) \Leftrightarrow ((V0x = \\ V1y) \vee (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V0x)\ V2s)))))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A-27a}). ((V0s = \\ (c\_2Epred\_set\_2EEMPTY\ A\_27a)) \vee (\exists V1x \in A\_27a. (\exists V2t \in \\ (2^{A-27a}). ((V0s = (ap\ (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V1x) \\ V2t)) \wedge (\neg (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V1x)\ V2t)))))))) \end{aligned} \quad (29)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((ap\ (c\_2Epred\_set\_2ECARD\ A\_27a) \\ (c\_2Epred\_set\_2EEMPTY\ A\_27a)) = c\_2Enum\_2E0) \quad (30)$$

**Theorem 1**

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A-27a}). ((p\ (ap \\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V0P)) \Rightarrow (\forall V1f \in (ty\_2Eextreal\_2Eextreal^{A-27a}). \\ (\forall V2x \in ty\_2Eextreal\_2Eextreal. ((\forall V3y \in A\_27a. ( \\ (ap\ V1f\ V3y) = V2x)) \Rightarrow ((ap\ (ap\ (c\_2Eextreal\_2EEXTREAL\_SUM\_IMAGE \\ A\_27a)\ V1f)\ V0P) = (ap\ (ap\ c\_2Eextreal\_2Eextreal\_mul\ (ap\ c\_2Eextreal\_2Eextreal\_of\_num \\ (ap\ (c\_2Epred\_set\_2ECARD\ A\_27a)\ V0P)))\ V2x)))))) \end{aligned}$$