

# thm\_2EfcP\_2Efinite\_\_sum (TMSvbD- pUf8agTCEiPhqxaHWLQKU1g9Tdo8E)

October 26, 2020

**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2ET` to be  $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (ap (ap (c_2Emin_2E_3D (2^{A-27a})) (\lambda V1x \in 2.V1x)) (\lambda V2x \in 2.V2x)))$

**Definition 4** We define `c_2Ebool_2EF` to be  $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$ .

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

**Definition 5** We define `c_2Epred__set_2EUNIV` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. c_2Ebool_2ET)$ .

**Definition 6** We define `c_2Ebool_2EIN` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. (\lambda V1f \in (2^{A-27a}). (ap V1f V0x)))$

**Definition 7** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p \Rightarrow q)$  of type  $\iota$ .

**Definition 8** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))))$

Let `c_2Epair_2EABS__prod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a.nonempty A. 27a \Rightarrow \forall A. 27b.nonempty A. 27b \Rightarrow c\_2Epair\_2EABS\_prod A. 27a A. 27b \in ((ty\_2Epair\_2Eprod A. 27a A. 27b)^{(2^{A-27b})^{A-27a}}) \tag{2}$$

**Definition 9** We define `c_2Epair_2E_2C` to be  $\lambda A. 27a : \iota. \lambda A. 27b : \iota. \lambda V0x \in A. 27a. \lambda V1y \in A. 27b. (ap (c_2Emin_2E_3D_3D_3E (c_2Ebool_2EF) (c_2Ebool_2EF)))$

Let `c_2Epred__set_2EGSPEC` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a.nonempty A. 27a \Rightarrow \forall A. 27b.nonempty A. 27b \Rightarrow c\_2Epred\_set\_2EGSPEC A. 27a A. 27b \in ((2^{A-27a})^{(ty\_2Epair\_2Eprod A. 27a 2)^{A-27b}}) \tag{3}$$

**Definition 10** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21) 2) (\lambda V2t \in$

**Definition 11** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap (c\_2Ebool\_2E\_21) 2) (\lambda V2t \in$

**Definition 12** We define  $c\_2Epred\_set\_2EIMAGE$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1s \in (2^{A\_27a}).(ap (c\_2Ebool\_2E\_21) 2) (\lambda V2t \in$

**Definition 13** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap (c\_2Ebool\_2E\_21) 2) (\lambda V2t \in$

**Definition 14** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2EF)$ .

**Definition 15** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).(ap (c\_2Ebool\_2E\_21) 2) (\lambda V2t \in$

**Definition 16** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\mathbf{if} (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 17** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap V0P (ap (c\_2Emin\_2E\_40) 2) (\lambda V2t \in$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Esum\_2Esum A0 A1) \quad (4)$$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Esum\_2EABS\_sum A\_27a A\_27b \in ((ty\_2Esum\_2Esum A\_27a A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \quad (5)$$

**Definition 18** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap (c\_2Esum\_2EABS\_sum) 2) (\lambda V2t \in$

**Definition 19** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap (c\_2Emin\_2E\_3D\_3D\_3E) V0t) c\_2Ebool\_2E\_21) 2) (\lambda V2t \in$

**Definition 20** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap (c\_2Esum\_2EABS\_sum) 2) (\lambda V2t \in$

Let  $c\_2Esum\_2EISR : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Esum\_2EISR A\_27a A\_27b \in (2^{(ty\_2Esum\_2Esum A\_27a A\_27b)}) \quad (6)$$

Let  $c\_2Esum\_2EISL : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Esum\_2EISL A\_27a A\_27b \in (2^{(ty\_2Esum\_2Esum A\_27a A\_27b)}) \quad (7)$$

Assume the following.

$$True \quad (8)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (9)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (10)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty \ A\_27a \Rightarrow \forall A\_27b.nonempty \ A\_27b \Rightarrow ( \\ \forall V0t1 \in A\_27a. (\forall V1t2 \in A\_27b. ((ap (\lambda V2x \in A\_27b. \\ V0t1) V1t2) = V0t1))) \end{aligned} \quad (11)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in \\ A\_27a. (p V0t)) \Leftrightarrow (p V0t))) \quad (12)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0t \in 2. ((\exists V1x \in \\ A\_27a. (p V0t)) \Leftrightarrow (p V0t))) \quad (13)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (14)$$

Assume the following.

$$\begin{aligned} ((\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (15)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow \\ True)) \quad (16)$$

Assume the following.

$$\forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in \\ A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (17)$$

Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty \ A\_27a \Rightarrow \forall A\_27b.nonempty \ A\_27b \Rightarrow ( \\ \forall V0f \in (A\_27b^{A\_27a}). (\forall V1g \in (A\_27b^{A\_27a}). ((V0f = \\ V1g) \Leftrightarrow (\forall V2x \in A\_27a. ((ap V0f V2x) = (ap V1g V2x)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ p V0t)))))) \end{aligned} \quad (19)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0a \in A\_27a. (\exists V1x \in A\_27a. (V1x = V0a))) \quad (20)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0x \in A\_27a. (\forall V1y \in A\_27b. (\forall V2a \in A\_27a. (\forall V3b \in \\ & \quad A\_27b. (((ap\ (ap\ (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V0x)\ V1y) = (ap\ (ap \\ & \quad (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b))))))))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}). (\forall V1x \in \\ & \quad A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V1x)\ V0P)) \Leftrightarrow (p\ (ap\ V0P\ V1x)))))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & \quad (2^{A\_27a}). ((V0s = V1t) \Leftrightarrow (\forall V2x \in A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\ & \quad A\_27a)\ V2x)\ V0s)) \Leftrightarrow (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V2x)\ V1t))))))))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in ((ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}). (\forall V1v \in \\ & \quad A\_27a. ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V1v)\ (ap\ (c\_2Epred\_set\_2EGSPEC \\ & \quad A\_27a\ A\_27b)\ V0f))) \Leftrightarrow (\exists V2x \in A\_27b. ((ap\ (ap\ (c\_2Epair\_2E\_2C \\ & \quad A\_27a\ 2)\ V1v)\ c\_2Ebool\_2ET) = (ap\ V0f\ V2x))))))))) \end{aligned} \quad (24)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A\_27a)\ V0x)\ (c\_2Epred\_set\_2EUNIV\ A\_27a)))) \quad (25)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}). (\forall V1t \in \\ & \quad (2^{A\_27a}). ((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ (ap\ (ap\ (c\_2Epred\_set\_2EUNION \\ & \quad A\_27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V0s)) \wedge \\ & \quad (p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ V1t))))))))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \quad \forall V0f \in (A\_27b^{A\_27a}). ((\forall V1x \in A\_27a. (\forall V2y \in \\ & \quad A\_27a. (((ap\ V0f\ V1x) = (ap\ V0f\ V2y)) \Leftrightarrow (V1x = V2y)))))) \Rightarrow (\forall V3s \in \\ & \quad (2^{A\_27a}). ((p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27b)\ (ap\ (ap\ (c\_2Epred\_set\_2EIMAGE \\ & \quad A\_27a\ A\_27b)\ V0f)\ V3s))) \Leftrightarrow (p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ \\ & \quad V3s))))))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & (\forall V0y \in A\_27a. (\forall V1x \in A\_27a. ((ap\ (c\_2Esum\_2EINL \\ & A\_27a\ A\_27b)\ V1x) = (ap\ (c\_2Esum\_2EINL\ A\_27a\ A\_27b)\ V0y))) \Leftrightarrow (V1x = \\ & V0y))) \wedge (\forall V2y \in A\_27b. (\forall V3x \in A\_27b. ((ap\ (c\_2Esum\_2EINR \\ & A\_27a\ A\_27b)\ V3x) = (ap\ (c\_2Esum\_2EINR\ A\_27a\ A\_27b)\ V2y))) \Leftrightarrow (V3x = \\ & V2y)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0ss \in (ty\_2Esum\_2Esum\ A\_27a\ A\_27b). ((\exists V1x \in A\_27a. \\ & (V0ss = (ap\ (c\_2Esum\_2EINL\ A\_27a\ A\_27b)\ V1x))) \vee (\exists V2y \in A\_27b. \\ & (V0ss = (ap\ (c\_2Esum\_2EINR\ A\_27a\ A\_27b)\ V2y)))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0x \in A\_27a. (\forall V1y \in A\_27b. (\neg((ap\ (c\_2Esum\_2EINL \\ & A\_27a\ A\_27b)\ V0x) = (ap\ (c\_2Esum\_2EINR\ A\_27a\ A\_27b)\ V1y)))))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & (\forall V0x \in A\_27a. (p\ (ap\ (c\_2Esum\_2EISL\ A\_27a\ A\_27b)\ (ap\ (c\_2Esum\_2EINL \\ & A\_27a\ A\_27b)\ V0x)))) \wedge (\forall V1y \in A\_27b. (\neg(p\ (ap\ (c\_2Esum\_2EISL \\ & A\_27a\ A\_27b)\ (ap\ (c\_2Esum\_2EINR\ A\_27a\ A\_27b)\ V1y)))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & (\forall V0x \in A\_27b. (p\ (ap\ (c\_2Esum\_2EISR\ A\_27a\ A\_27b)\ (ap\ (c\_2Esum\_2EINR \\ & A\_27a\ A\_27b)\ V0x)))) \wedge (\forall V1y \in A\_27a. (\neg(p\ (ap\ (c\_2Esum\_2EISR \\ & A\_27a\ A\_27b)\ (ap\ (c\_2Esum\_2EINL\ A\_27a\ A\_27b)\ V1y)))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0x \in (ty\_2Esum\_2Esum\ A\_27a\ A\_27b). ((p\ (ap\ (c\_2Esum\_2EISL \\ & A\_27a\ A\_27b)\ V0x)) \vee (p\ (ap\ (c\_2Esum\_2EISR\ A\_27a\ A\_27b)\ V0x)))) \end{aligned} \quad (33)$$

**Theorem 1**

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & (p\ (ap\ (c\_2Epred\_set\_2EFINITE\ (ty\_2Esum\_2Esum\ A\_27a\ A\_27b)) \\ & (c\_2Epred\_set\_2EUNIV\ (ty\_2Esum\_2Esum\ A\_27a\ A\_27b)))) \Leftrightarrow ((p\ ( \\ & ap\ (c\_2Epred\_set\_2EFINITE\ A\_27a)\ (c\_2Epred\_set\_2EUNIV\ A\_27a))) \wedge \\ & (p\ (ap\ (c\_2Epred\_set\_2EFINITE\ A\_27b)\ (c\_2Epred\_set\_2EUNIV \\ & A\_27b)))))) \end{aligned}$$