

thm\_2Efinite\_\_map\_2EDOMSUB\_\_SUBMAP  
(TMNk81w5H4AuJnrgNpWJEiHHP9D5g2kEHZv)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2))) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x)$

Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \tag{1}$$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \tag{2}$$

Let  $ty\_2Efinite\_map\_2Efmap : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Efinite\_map\_2Efmap\ A0\ A1) \tag{3}$$

Let  $c\_2Efinite\_map\_2Efmap\_REP : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Efinite\_map\_2Efmap\_REP\ A\_27a\ A\_27b \in (((ty\_2Esum\_2Esum\ A\_27b\ ty\_2Eone\_2Eone)^{A\_27a})(ty\_2Efinite\_map\_2Efmap\ A\_27a\ A\_27b)) \tag{4}$$

Let  $c\_2Esum\_2EOUTL : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EOUTL\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Esum\_2Esum\ A\_27a\ A\_27b)}) \tag{5}$$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})))$

**Definition 4** We define  $c\_2Efinite\_map\_2EFAPPLY$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (ty\_2Efinite\_map\_2Efmap$

Let  $c\_2Esum\_2EISL : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EISL \\ A\_27a\ A\_27b \in (2^{(ty\_2Esum\_2Esum\ A\_27a\ A\_27b)}) \end{aligned} \quad (6)$$

**Definition 5** We define  $c\_2Efinite\_map\_2EFDOM$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (ty\_2Efinite\_map\_2EFDOM\ A\_27a\ A\_27b)$

**Definition 6** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap\ V1f\ V0x)))$

**Definition 7** We define  $c\_2Emin\_2E3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o\ (p\ P \Rightarrow p\ Q)$  of type  $\iota$ .

**Definition 8** We define  $c\_2Ebool\_2E2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E21\ 2)\ (\lambda V2t \in 2.V0t))))$

**Definition 9** We define  $c\_2Efinite\_map\_2ESUBMAP$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (ty\_2Efinite\_map\_2ESUBMAP\ A\_27a\ A\_27b)$

**Definition 10** We define  $c\_2Emin\_2E40$  to be  $\lambda A.\lambda P \in 2^A.if\ (\exists x \in A.p\ (ap\ P\ x))\ then\ (the\ (\lambda x.x \in A)\ P)$  of type  $\iota \Rightarrow \iota$ .

**Definition 11** We define  $c\_2Eone\_2Eone$  to be  $(ap\ (c\_2Emin\_2E40\ ty\_2Eone\_2Eone)\ (\lambda V0x \in ty\_2Eone\_2Eone))$

**Definition 12** We define  $c\_2Ebool\_2E2F$  to be  $(ap\ (c\_2Ebool\_2E21\ 2)\ (\lambda V0t \in 2.V0t))$ .

**Definition 13** We define  $c\_2Ebool\_2E7E$  to be  $(\lambda V0t \in 2.(ap\ (ap\ c\_2Emin\_2E3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E2F))$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum \\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \end{aligned} \quad (7)$$

**Definition 14** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap\ (c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b)\ V0e)$

Let  $c\_2Efinite\_map\_2Efmap\_ABS : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Efinite\_map\_2Efmap\_ABS \\ A\_27a\ A\_27b \in ((ty\_2Efinite\_map\_2Efmap\ A\_27a\ A\_27b)^{(ty\_2Esum\_2Esum\ A\_27b\ ty\_2Eone\_2Eone)^{A\_27a}}) \end{aligned} \quad (8)$$

**Definition 15** We define  $c\_2Efinite\_map\_2EFEMPTY$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(ap\ (c\_2Efinite\_map\_2Efmap\_ABS\ A\_27a\ A\_27b)\ ty\_2Eone\_2Eone)$

**Definition 16** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.(ap\ (c\_2Ebool\_2E2F)\ V2t2))))$

**Definition 17** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2E2F)$ .

**Definition 18** We define  $c\_2Ebool\_2E5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E21\ 2)\ (\lambda V2t \in 2.V0t))))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (9)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (10)$$

**Definition 19** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b)\ V0x\ V1y)$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \quad (11)$$

**Definition 20** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27a)\ V0x\ V1s)$

**Definition 21** We define  $c\_2Epred\_set\_2EUNIV$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2ET)$ .

**Definition 22** We define  $c\_2Epred\_set\_2EDIFF$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27a)\ V0s\ V1t)$

**Definition 23** We define  $c\_2Epred\_set\_2Ecompl$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27a)\ V0P))$

Let  $c\_2Efinite\_map\_2EDRESTRICT : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Efinite\_map\_2EDRESTRICT\ A\_27a\ A\_27b \in (((ty\_2Efinite\_map\_2Efm\ A\_27a\ A\_27b)^{(2^{A\_27a})})^{(ty\_2Efinite\_map\_2Efm\ A\_27a\ A\_27b)}) \quad (12)$$

**Definition 24** We define  $c\_2Efinite\_map\_2Efdmsub$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0fm \in (ty\_2Efinite\_map\_2EDRESTRICT\ A\_27a\ A\_27b)$

**Definition 25** We define  $c\_2Epred\_set\_2EDELETE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1x \in A\_27a.(ap\ (ap\ (c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27a)\ V0s)\ V1x)$

Assume the following.

$$True \quad (13)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (14)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \quad (15)$$

Assume the following.

$$(\forall V0t \in 2.((p V0t) \vee (\neg(p V0t)))) \quad (16)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow \text{False}))) \quad (17)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((\text{True} \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge \text{True}) \Leftrightarrow \\ & (p V0t)) \wedge (((\text{False} \wedge (p V0t)) \Leftrightarrow \text{False}) \wedge (((p V0t) \wedge \text{False}) \Leftrightarrow \text{False}) \wedge \\ & (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (18)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((\text{True} \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow \text{True}) \Leftrightarrow \\ & \text{True}) \wedge (((\text{False} \Rightarrow (p V0t)) \Leftrightarrow \text{True}) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow \text{True}) \wedge (( \\ & (p V0t) \Rightarrow \text{False}) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (19)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg \text{True}) \Leftrightarrow \text{False}) \wedge \\ & ((\neg \text{False}) \Leftrightarrow \text{True}))) \end{aligned} \quad (20)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow \text{True})) \quad (21)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (22)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((\text{True} \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow \text{True}) \Leftrightarrow \\ & (p V0t)) \wedge (((\text{False} \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow \text{False}) \Leftrightarrow (\neg( \\ & p V0t)))))) \end{aligned} \quad (23)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0t1 \in A\_27a.(\forall V1t2 \in \\ & A\_27a.(((\text{ap } (\text{ap } (\text{ap } (\text{c\_2Ebool\_2ECOND } A\_27a) \text{ c\_2Ebool\_2ET}) V0t1) \\ & V1t2) = V0t1) \wedge ((\text{ap } (\text{ap } (\text{ap } (\text{c\_2Ebool\_2ECOND } A\_27a) \text{ c\_2Ebool\_2EF}) \\ & V0t1) V1t2) = V1t2)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow \\ & ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \end{aligned} \quad (25)$$

Assume the following.

$$2.(((\forall V0x \in 2.(\forall V1x_{27} \in 2.(\forall V2y \in 2.(\forall V3y_{27} \in 2.(((p V0x) \Leftrightarrow (p V1x_{27})) \wedge ((p V1x_{27}) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_{27})))))) \Rightarrow ((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_{27}) \Rightarrow (p V3y_{27})))))) \Rightarrow (26)$$

Assume the following.

$$\forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow (\forall V0fm \in (ty\_2Efinite\_map\_2Efm\ A_{.27a}\ A_{.27b}).(\forall V1k \in A_{.27a}.((ap\ (c\_2Efinite\_map\_2EFDOM\ A_{.27a}\ A_{.27b})\ (ap\ (ap\ (c\_2Efinite\_map\_2Efdomsub\ A_{.27a}\ A_{.27b})\ V0fm)\ V1k)) = (ap\ (ap\ (c\_2Epred\_set\_2EDELETE\ A_{.27a})\ (ap\ (c\_2Efinite\_map\_2EFDOM\ A_{.27a}\ A_{.27b})\ V0fm))\ V1k)))) (27)$$

Assume the following.

$$\forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow (\forall V0fm \in (ty\_2Efinite\_map\_2Efm\ A_{.27a}\ A_{.27b}).(\forall V1k1 \in A_{.27a}.(\forall V2k2 \in A_{.27a}.((ap\ (ap\ (c\_2Efinite\_map\_2EFAPPLY\ A_{.27a}\ A_{.27b})\ (ap\ (ap\ (c\_2Efinite\_map\_2Efdomsub\ A_{.27a}\ A_{.27b})\ V0fm)\ V1k1))\ V2k2) = (ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A_{.27b})\ (ap\ (ap\ (c\_2Emin\_2E\_3D\ A_{.27a})\ V1k1)\ V2k2))\ (ap\ (ap\ (c\_2Efinite\_map\_2EFAPPLY\ A_{.27a}\ A_{.27b})\ (c\_2Efinite\_map\_2EFEMPTY\ A_{.27a}\ A_{.27b})\ V2k2))\ (ap\ (ap\ (c\_2Efinite\_map\_2EFAPPLY\ A_{.27a}\ A_{.27b})\ V0fm)\ V2k2)))))) (28)$$

Assume the following.

$$\forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow (\forall V0s \in (2^{A_{.27a}}).(\forall V1x \in A_{.27a}.(\forall V2y \in A_{.27a}.((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A_{.27a})\ V1x)\ (ap\ (ap\ (c\_2Epred\_set\_2EDELETE\ A_{.27a})\ V0s)\ V2y))) \Leftrightarrow ((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A_{.27a})\ V1x)\ V0s)) \wedge (\neg(V1x = V2y)))))) (29)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) (30)$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) (31)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow (((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) (32)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(\neg(p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False)))))) (33)$$

Assume the following.

$$(\forall V0A \in 2.((\neg(p V0A)) \Rightarrow False) \Rightarrow ((p V0A) \Rightarrow False) \Rightarrow False)) \quad (34)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee (\neg( \\ & p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\ & ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow ( \\ & (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ( \\ & \neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \end{aligned} \quad (36)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p)))) \quad (37)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q)))) \quad (38)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V0p)))) \quad (39)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))) \quad (40)$$

Assume the following.

$$(\forall V0p \in 2.((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \quad (41)$$

### Theorem 1

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow ( \\ & \forall V0f \in (ty\_2Efinite\_map\_2E fmap A\_27a A\_27b).(\forall V1g \in \\ & (ty\_2Efinite\_map\_2E fmap A\_27a A\_27b).(\forall V2x \in A\_27a.( \\ & ((p (ap (ap (c\_2Efinite\_map\_2ESUBMAP A\_27a A\_27b) V0f) V1g)) \wedge \\ & (\neg(p (ap (ap (c\_2Ebool\_2EIN A\_27a) V2x) (ap (c\_2Efinite\_map\_2EFDOM \\ & A\_27a A\_27b) V0f)))))) \Rightarrow (p (ap (ap (c\_2Efinite\_map\_2ESUBMAP A\_27a \\ & A\_27b) V0f) (ap (ap (c\_2Efinite\_map\_2Efdomsub A\_27a A\_27b) V1g) \\ & V2x)))))) \end{aligned}$$