

# thm\_2Efinite\_\_map\_2EDRESTRICT\_\_SUBMAP (TMGvqR5fHquroYYKjuTyDTadZjoFCzEKZYC)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \quad (1)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (2)$$

**Definition 7** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Ebool\_2E\_2F$

Let  $ty\_2Efinite\_map\_2Eefmap : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Efinite\_map\_2Eefmap A0 A1) \quad (3)$$

Let  $c\_2Efinite\_map\_2EFUPDATE : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$A\_27a A\_27b \in (((ty\_2Efinite\_map\_2Eefmap A\_27a A\_27b)^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}(ty\_2Efinite\_map\_2EFUPDATE A\_27a A\_27b))^{(2^{A\_27b})^{A\_27a}}) \quad (4)$$

**Definition 8** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_7E))$ . Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \quad (5)$$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \quad (6)$$

Let  $c\_2Efinite\_map\_2E\_fmap\_REP : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Efinite\_map\_2E\_fmap\_REP\ A\_27a\ A\_27b \in (((ty\_2Esum\_2Esum\ A\_27b\ ty\_2Eone\_2Eone)^{A\_27a})^{(ty\_2Efinite\_map\_2E\_fmap\ A\_27a\ A\_27b)}) \quad (7)$$

Let  $c\_2Esum\_2EOUTL : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EOUTL\ A\_27a\ A\_27b \in (A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ A\_27b)} \quad (8)$$

**Definition 9** We define  $c\_2Efinite\_map\_2E\_FAPPLY$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (ty\_2Efinite\_map\_2E\_fmap\ A\_27a\ A\_27b)$

Let  $c\_2Esum\_2EISL : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EISL\ A\_27a\ A\_27b \in (2^{(ty\_2Esum\_2Esum\ A\_27a\ A\_27b)}) \quad (9)$$

**Definition 10** We define  $c\_2Efinite\_map\_2E\_FDOM$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (ty\_2Efinite\_map\_2E\_fmap\ A\_27a\ A\_27b)$

**Definition 11** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap\ V1f\ V0x)))$

**Definition 12** We define  $c\_2Efinite\_map\_2E\_SUBMAP$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (ty\_2Efinite\_map\_2E\_fmap\ A\_27a\ A\_27b)$

**Definition 13** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p (ap\ P\ x)) \text{ then } (the\ (\lambda x.x \in A \wedge p\ x)) \text{ of type } \iota \Rightarrow \iota.$

**Definition 14** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.(\lambda V3t3 \in 2^{A\_27a}.(ap\ V3t3\ V2t2)))))$

**Definition 15** We define  $c\_2Eone\_2Eone$  to be  $(ap (c\_2Emin\_2E\_40\ ty\_2Eone\_2Eone)) (\lambda V0x \in ty\_2Eone\_2Eone)$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \quad (10)$$

**Definition 16** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap (c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b))$

Let  $c\_2Efinite\_map\_2Efmap\_ABS : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Efinite\_map\_2Efmap\_ABS \\ & A\_27a\ A\_27b \in ((ty\_2Efinite\_map\_2Efmap\ A\_27a\ A\_27b)^{(ty\_2Esum\_2Esum\ A\_27b\ ty\_2Eone\_2Eone)^{A\_27a}}) \end{aligned} \quad (11)$$

**Definition 17** We define  $c\_2Efinite\_map\_2EFEMPTY$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(ap\ (c\_2Efinite\_map\_2E$

Let  $c\_2Efinite\_map\_2EDRESTRICT : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Efinite\_map\_2EDRESTRICT \\ & A\_27a\ A\_27b \in (((ty\_2Efinite\_map\_2Efmap\ A\_27a\ A\_27b)^{(2^{A\_27a}}))^{(ty\_2Efinite\_map\_2Efmap\ A\_27a\ A\_27b)}) \end{aligned} \quad (12)$$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC \\ & A\_27a\ A\_27b \in ((2^{A\_27a})^{((ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b})}) \end{aligned} \quad (13)$$

**Definition 18** We define  $c\_2Epred\_set\_2EINTER$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap\ (c\_2$

**Definition 19** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2.$

**Definition 20** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap\ (c\_2$

Assume the following.

$$True \quad (14)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in \\ & A\_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \end{aligned} \quad (16)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (( \\ & (p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))) \end{aligned} \quad (17)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow \\ & True)) \end{aligned} \quad (18)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. (\forall V1y \in A\_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (19)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg(p\ V0t)))))) \quad (20)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p\ V0t1) \Rightarrow ((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \quad (21)$$

Assume the following.

$$(\forall V0x \in 2. (\forall V1x\_27 \in 2. (\forall V2y \in 2. (\forall V3y\_27 \in 2. (((p\ V0x) \Leftrightarrow (p\ V1x\_27)) \wedge ((p\ V1x\_27) \Rightarrow ((p\ V2y) \Leftrightarrow (p\ V3y\_27)))) \Rightarrow (((p\ V0x) \Rightarrow (p\ V2y)) \Leftrightarrow ((p\ V1x\_27) \Rightarrow (p\ V3y\_27)))))) \quad (22)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0P \in 2. (\forall V1Q \in 2. (\forall V2x \in A\_27a. (\forall V3x\_27 \in A\_27a. (\forall V4y \in A\_27a. (\forall V5y\_27 \in A\_27a. (((p\ V0P) \Leftrightarrow (p\ V1Q)) \wedge ((p\ V1Q) \Rightarrow (V2x = V3x\_27)) \wedge ((\neg(p\ V1Q)) \Rightarrow (V4y = V5y\_27)))) \Rightarrow ((ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ V0P)\ V2x)\ V4y) = (ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ V1Q)\ V3x\_27)\ V5y\_27)))))) \quad (23)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((\forall V0t1 \in A\_27a. (\forall V1t2 \in A\_27a. ((ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ c\_2Ebool\_2ET)\ V0t1)\ V1t2) = V0t1))) \wedge (\forall V2t1 \in A\_27a. (\forall V3t2 \in A\_27a. ((ap\ (ap\ (c\_2Ebool\_2ECOND\ A\_27a)\ c\_2Ebool\_2EF)\ V2t1)\ V3t2) = V3t2)))) \quad (24)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0f \in (ty\_2Efinite\_map\_2Efmap\ A\_27a\ A\_27b). (\forall V1a \in A\_27a. (\forall V2b \in A\_27b. ((ap\ (c\_2Efinite\_map\_2EFDOM\ A\_27a\ A\_27b)\ (ap\ (ap\ (c\_2Efinite\_map\_2EFUPDATE\ A\_27a\ A\_27b)\ V0f)\ (ap\ (ap\ (c\_2Epair\_2E\_2C\ A\_27a\ A\_27b)\ V1a)\ V2b))) = (ap\ (ap\ (c\_2Epred\_set\_2EINSERT\ A\_27a)\ V1a)\ (ap\ (c\_2Efinite\_map\_2EFDOM\ A\_27a\ A\_27b)\ V0f)))))) \quad (25)$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\
& \quad \forall V0P \in (2^{(ty\_2Efinite\_map\_2E fmap\ A.27a\ A.27b)}).((p \\
& \quad (ap\ V0P\ (c\_2Efinite\_map\_2EFEMPTY\ A.27a\ A.27b))) \wedge (\forall V1f \in \\
& \quad (ty\_2Efinite\_map\_2E fmap\ A.27a\ A.27b).((p\ (ap\ V0P\ V1f)) \Rightarrow (\forall V2x \in \\
& \quad A.27a.(\forall V3y \in A.27b.((\neg(p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A.27a) \\
& \quad V2x)\ (ap\ (c\_2Efinite\_map\_2EFDOM\ A.27a\ A.27b)\ V1f)))))) \Rightarrow (p\ (ap\ V0P \\
& \quad (ap\ (ap\ (c\_2Efinite\_map\_2EFUPDATE\ A.27a\ A.27b)\ V1f)\ (ap\ (ap\ (c\_2Epair\_2E\_2C \\
& \quad A.27a\ A.27b)\ V2x)\ V3y)))))))))) \Rightarrow (\forall V4f \in (ty\_2Efinite\_map\_2E fmap \\
& \quad A.27a\ A.27b).(p\ (ap\ V0P\ V4f))))))
\end{aligned} \tag{26}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\
& \quad \forall V0f \in (ty\_2Efinite\_map\_2E fmap\ A.27a\ A.27b).(p\ (ap\ (ap \\
& \quad (c\_2Efinite\_map\_2ESUBMAP\ A.27a\ A.27b)\ (c\_2Efinite\_map\_2EFEMPTY \\
& \quad A.27a\ A.27b))\ V0f)))
\end{aligned} \tag{27}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\
& \quad \forall V0f \in (ty\_2Efinite\_map\_2E fmap\ A.27a\ A.27b).(\forall V1r \in \\
& \quad (2^{A.27a}).(((ap\ (c\_2Efinite\_map\_2EFDOM\ A.27a\ A.27b)\ (ap\ (ap \\
& \quad (c\_2Efinite\_map\_2EDRESTRICT\ A.27a\ A.27b)\ V0f)\ V1r)) = (ap\ (ap \\
& \quad (c\_2Epred\_set\_2EINTER\ A.27a)\ (ap\ (c\_2Efinite\_map\_2EFDOM\ A.27a \\
& \quad A.27b)\ V0f))\ V1r)) \wedge (\forall V2x \in A.27a.((ap\ (ap\ (c\_2Efinite\_map\_2EFAPPLY \\
& \quad A.27a\ A.27b)\ (ap\ (ap\ (c\_2Efinite\_map\_2EDRESTRICT\ A.27a\ A.27b) \\
& \quad V0f)\ V1r))\ V2x) = (ap\ (ap\ (ap\ (c\_2Ebool\_2ECOND\ A.27b)\ (ap\ (ap\ (c\_2Ebool\_2EIN \\
& \quad A.27a)\ V2x)\ (ap\ (ap\ (c\_2Epred\_set\_2EINTER\ A.27a)\ (ap\ (c\_2Efinite\_map\_2EFDOM \\
& \quad A.27a\ A.27b)\ V0f))\ V1r)))\ (ap\ (ap\ (c\_2Efinite\_map\_2EFAPPLY\ A.27a \\
& \quad A.27b)\ V0f)\ V2x))\ (ap\ (ap\ (c\_2Efinite\_map\_2EFAPPLY\ A.27a\ A.27b) \\
& \quad (c\_2Efinite\_map\_2EFEMPTY\ A.27a\ A.27b))\ V2x))))))
\end{aligned} \tag{28}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\
& \quad \forall V0r \in (2^{A.27a}).((ap\ (ap\ (c\_2Efinite\_map\_2EDRESTRICT \\
& \quad A.27a\ A.27b)\ (c\_2Efinite\_map\_2EFEMPTY\ A.27a\ A.27b))\ V0r) = (c\_2Efinite\_map\_2EFEMPTY \\
& \quad A.27a\ A.27b)))
\end{aligned} \tag{29}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0s \in (2^{A.27a}).(\forall V1t \in \\
& \quad (2^{A.27a}).(\forall V2x \in A.27a.((p\ (ap\ (ap\ (c\_2Ebool\_2EIN\ A.27a) \\
& \quad V2x)\ (ap\ (c\_2Epred\_set\_2EINTER\ A.27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap \\
& \quad (ap\ (c\_2Ebool\_2EIN\ A.27a)\ V2x)\ V0s)) \wedge (p\ (ap\ (ap\ (c\_2Ebool\_2EIN \\
& \quad A.27a)\ V2x)\ V1t))))))
\end{aligned} \tag{30}$$

Assume the following.

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a. (\forall V1y \in \\
& A.27a. (\forall V2s \in (2^{A.27a}). ((p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a) \\
& V0x)\ (ap\ (ap\ (c.2Epred\_set.2EINSERT\ A.27a)\ V1y)\ V2s)))) \Leftrightarrow ((V0x = \\
& V1y) \vee (p\ (ap\ (ap\ (c.2Ebool.2EIN\ A.27a)\ V0x)\ V2s))))))
\end{aligned} \tag{31}$$

**Theorem 1**

$$\begin{aligned}
& \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow ( \\
& \forall V0f \in (ty\_2Efinite\_map.2Efmap\ A.27a\ A.27b). (\forall V1r \in \\
& (2^{A.27a}). (p\ (ap\ (ap\ (c.2Efinite\_map.2ESUBMAP\ A.27a\ A.27b)\ ( \\
& ap\ (ap\ (c.2Efinite\_map.2EDRESTRICT\ A.27a\ A.27b)\ V0f)\ V1r))\ V0f))))
\end{aligned}$$