

thm_2Efinite_map_2EDRESTRICT_SUBSET_SUBMAP (TMEob4soUiuAcvFbouWnhK4H3ri42t3tK2e)

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Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 4 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 6 We define $c_2Ebool_2E_27E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \tag{1}$$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \tag{2}$$

Let $ty_2Efinite_map_2Efmap : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Efinite_map_2Efmap\ A0\ A1) \tag{3}$$

Let $c_2Efinite_map_2Efmap_REP : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efinite_map_2Efmap_REP\ A_27a\ A_27b \in (((ty_2Esum_2Esum\ A_27b\ ty_2Eone_2Eone)^{A_27a})(ty_2Efinite_map_2Efmap\ A_27a\ A_27b)) \tag{4}$$

Let $c_2Esum_2EOUTL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EOUTL \\ A_27a\ A_27b \in (A_27a^{(ty_2Esum_2Esum\ A_27a\ A_27b)}) \end{aligned} \quad (5)$$

Definition 7 We define $c_2Efinite_map_2EFAPPLY$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (ty_2Efinite_map_2EFAPPLY)$

Let $c_2Esum_2EISL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EISL \\ A_27a\ A_27b \in (2^{(ty_2Esum_2Esum\ A_27a\ A_27b)}) \end{aligned} \quad (6)$$

Definition 8 We define $c_2Efinite_map_2EFDOM$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (ty_2Efinite_map_2EFDOM)$

Definition 9 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(\text{ap } V1f\ V0x)))$

Definition 10 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(\text{ap } (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in 2))))$

Definition 11 We define $c_2Efinite_map_2ESUBMAP$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (ty_2Efinite_map_2ESUBMAP)$

Definition 12 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p\ (\text{ap } P\ x)) \text{ then } (the\ (\lambda x.x \in A \wedge P\ x))$ of type $\iota \Rightarrow \iota$.

Definition 13 We define c_2Eone_2Eone to be $(\text{ap } (c_2Emin_2E_40\ ty_2Eone_2Eone)\ (\lambda V0x \in ty_2Eone_2Eone))$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum \\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \end{aligned} \quad (7)$$

Definition 14 We define c_2Esum_2EINR to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27b.(\text{ap } (c_2Esum_2EABS_sum))$

Let $c_2Efinite_map_2Efmap_ABS : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efinite_map_2Efmap_ABS \\ A_27a\ A_27b \in ((ty_2Efinite_map_2Efmap\ A_27a\ A_27b)^{((ty_2Esum_2Esum\ A_27b\ ty_2Eone_2Eone)^{A_27a})}) \end{aligned} \quad (8)$$

Definition 15 We define $c_2Efinite_map_2EFEMPTY$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(\text{ap } (c_2Efinite_map_2EFEMPTY))$

Definition 16 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.(\text{ap } (c_2Ebool_2ECOND))))$

Let $c_2Efinite_map_2EDRESTRICT : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efinite_map_2EDRESTRICT \\ A_27a\ A_27b \in (((ty_2Efinite_map_2Efmap\ A_27a\ A_27b)^{(2^{A_27a})})^{(ty_2Efinite_map_2Efmap\ A_27a\ A_27b)}) \end{aligned} \quad (9)$$

Definition 17 We define $c_2Epred_set_2ESUBSET$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap (c_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota)$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \quad (10)$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epair_2EABS_prod A_27a A_27b \in ((ty_2Epair_2Eprod A_27a A_27b)^{(2^{A_27b})^{A_27a}}) \quad (11)$$

Definition 18 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap (c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota)$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epred_set_2EGSPEC A_27a A_27b \in ((2^{A_27a})^{((ty_2Epair_2Eprod A_27a 2)^{A_27b})}) \quad (12)$$

Definition 19 We define $c_2Epred_set_2EINTER$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap (c_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota)$ be given. Assume the following.

$$True \quad (13)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A_27a.(p V0t)) \Leftrightarrow (p V0t))) \quad (14)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((\\ & (p V0t) \Rightarrow False) \Leftrightarrow \neg (p V0t)))))) \end{aligned} \quad (16)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0x \in A_27a.((V0x = V0x) \Leftrightarrow True)) \quad (17)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0x \in A_27a.(\forall V1y \in A_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (18)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow \neg(p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow \neg(\\
& p \ V0t))))))
\end{aligned} \tag{19}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p \ V0t1) \Rightarrow \\
& ((p \ V1t2) \Rightarrow (p \ V2t3))) \Leftrightarrow (((p \ V0t1) \wedge (p \ V1t2)) \Rightarrow (p \ V2t3))))))
\end{aligned} \tag{20}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in 2.(\forall V1x_{.27} \in 2.(\forall V2y \in 2.(\forall V3y_{.27} \in \\
& 2.(((p \ V0x) \Leftrightarrow (p \ V1x_{.27})) \wedge ((p \ V1x_{.27}) \Rightarrow ((p \ V2y) \Leftrightarrow (p \ V3y_{.27})))) \Rightarrow \\
& (((p \ V0x) \Rightarrow (p \ V2y)) \Leftrightarrow ((p \ V1x_{.27}) \Rightarrow (p \ V3y_{.27}))))))
\end{aligned} \tag{21}$$

Assume the following.

$$\begin{aligned}
& \forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow (\forall V0P \in 2.(\forall V1Q \in 2. \\
& (\forall V2x \in A_{.27a}.(\forall V3x_{.27} \in A_{.27a}.(\forall V4y \in A_{.27a}. \\
& (\forall V5y_{.27} \in A_{.27a}.(((p \ V0P) \Leftrightarrow (p \ V1Q)) \wedge (((p \ V1Q) \Rightarrow (V2x = V3x_{.27})) \wedge \\
& (\neg(p \ V1Q)) \Rightarrow (V4y = V5y_{.27})))) \Rightarrow ((ap \ (ap \ (ap \ (c_{.2Ebool_2ECOND} \ A_{.27a}) \\
& V0P) \ V2x) \ V4y) = (ap \ (ap \ (ap \ (c_{.2Ebool_2ECOND} \ A_{.27a}) \ V1Q) \ V3x_{.27}) \\
& V5y_{.27}))))))
\end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned}
& \forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow ((\forall V0t1 \in A_{.27a}.(\forall V1t2 \in \\
& A_{.27a}.((ap \ (ap \ (ap \ (c_{.2Ebool_2ECOND} \ A_{.27a}) \ c_{.2Ebool_2ET} \ V0t1) \\
& V1t2) = V0t1))) \wedge (\forall V2t1 \in A_{.27a}.(\forall V3t2 \in A_{.27a}.((ap \\
& (ap \ (ap \ (c_{.2Ebool_2ECOND} \ A_{.27a}) \ c_{.2Ebool_2EF} \ V2t1) \ V3t2) = V3t2))))))
\end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned}
& \forall A_{.27a}.nonempty \ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty \ A_{.27b} \Rightarrow (\\
& \forall V0f \in (ty_{.2Efinite_map_2E fmap} \ A_{.27a} \ A_{.27b}).(\forall V1r \in \\
& (2^{A_{.27a}}).(((ap \ (c_{.2Efinite_map_2EFDOM} \ A_{.27a} \ A_{.27b}) \ (ap \ (ap \\
& (c_{.2Efinite_map_2EDRESTRICT} \ A_{.27a} \ A_{.27b}) \ V0f) \ V1r)) = (ap \ (ap \\
& (c_{.2Epred_set_2EINTER} \ A_{.27a}) \ (ap \ (c_{.2Efinite_map_2EFDOM} \ A_{.27a} \\
& A_{.27b}) \ V0f)) \ V1r)) \wedge (\forall V2x \in A_{.27a}.((ap \ (ap \ (c_{.2Efinite_map_2EFAPPLY} \\
& A_{.27a} \ A_{.27b}) \ (ap \ (ap \ (c_{.2Efinite_map_2EDRESTRICT} \ A_{.27a} \ A_{.27b}) \\
& V0f) \ V1r)) \ V2x) = (ap \ (ap \ (ap \ (c_{.2Ebool_2ECOND} \ A_{.27b}) \ (ap \ (ap \ (c_{.2Ebool_2EIN} \\
& A_{.27a}) \ V2x) \ (ap \ (ap \ (c_{.2Epred_set_2EINTER} \ A_{.27a}) \ (ap \ (c_{.2Efinite_map_2EFDOM} \\
& A_{.27a} \ A_{.27b}) \ V0f)) \ V1r))) \ (ap \ (ap \ (c_{.2Efinite_map_2EFAPPLY} \ A_{.27a} \\
& A_{.27b}) \ V0f) \ V2x)) \ (ap \ (ap \ (c_{.2Efinite_map_2EFAPPLY} \ A_{.27a} \ A_{.27b}) \\
& (c_{.2Efinite_map_2EFEMPTY} \ A_{.27a} \ A_{.27b})) \ V2x))))))
\end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0s \in (2^{A_27a}). (\forall V1t \in \\
& (2^{A_27a}). (\forall V2x \in A_27a. ((p\ (ap\ (ap\ (c_2Ebool_2EIN\ A_27a) \\
& V2x)\ (ap\ (ap\ (c_2Epred_set_2EINTER\ A_27a)\ V0s)\ V1t))) \Leftrightarrow ((p\ (ap\ (25) \\
& (ap\ (c_2Ebool_2EIN\ A_27a)\ V2x)\ V0s)) \wedge (p\ (ap\ (ap\ (c_2Ebool_2EIN \\
& A_27a)\ V2x)\ V1t))))))
\end{aligned}$$

Theorem 1

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \forall V0s1 \in (2^{A_27a}). (\forall V1s2 \in (2^{A_27a}). (\forall V2f \in \\
& (ty_2Efinite_map_2Efmap\ A_27a\ A_27b). ((p\ (ap\ (ap\ (c_2Epred_set_2ESUBSET \\
& A_27a)\ V0s1)\ V1s2))) \Rightarrow (p\ (ap\ (ap\ (c_2Efinite_map_2ESUBMAP\ A_27a \\
& A_27b)\ (ap\ (ap\ (c_2Efinite_map_2EDRESTRICT\ A_27a\ A_27b)\ V2f) \\
& V0s1))\ (ap\ (ap\ (c_2Efinite_map_2EDRESTRICT\ A_27a\ A_27b)\ V2f) \\
& V1s2))))))
\end{aligned}$$