

thm_2Efinite__map_2EFMERGE__FEMPTY
(TM-
SuEg4dGGUc4Yyw7GBvCHqRTTgUU3Y4bv1)

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Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a})))$

Definition 4 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \tag{1}$$

Definition 5 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p$ of type $\iota \Rightarrow \iota$.

Definition 6 We define c_2Eone_2Eone to be $(ap (c_2Emin_2E_40 ty_2Eone_2Eone) (\lambda V0x \in ty_2Eone_2Eone$

Definition 7 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 8 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F$

Definition 9 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t)))$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \tag{2}$$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \tag{3}$$

Definition 10 We define c_Esum_2EINR to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0e \in A_27b. (ap (c_Esum_2EABS$

Let $ty_2Efinite_map_2Efmap : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Efinite_map_2Efmap A0 A1) \quad (4)$$

Let $c_2Efinite_map_2Efmap_ABS : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Efinite_map_2Efmap_ABS A_27a A_27b \in ((ty_2Efinite_map_2Efmap A_27a A_27b)^{(ty_2Esum_2Esum A_27b ty_2Eone_2Eone)^{A_27a}}) \quad (5)$$

Definition 11 We define $c_2Efinite_map_2EFEMPTY$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. (ap (c_2Efinite_map_2E$

Definition 12 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A_27a. (\lambda V2t2 \in A_27a. ($

Let $c_2Efinite_map_2Efmap_REP : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Efinite_map_2Efmap_REP A_27a A_27b \in (((ty_2Esum_2Esum A_27b ty_2Eone_2Eone)^{A_27a})^{(ty_2Efinite_map_2Efmap A_27a A_27b)}) \quad (6)$$

Let $c_2Esum_2EOUTL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Esum_2EOUTL A_27a A_27b \in (A_27a^{(ty_2Esum_2Esum A_27a A_27b)}) \quad (7)$$

Definition 13 We define $c_2Efinite_map_2EFAPPLY$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0f \in (ty_2Efinite_map$

Let $c_2Efinite_map_2EFMERGE : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Efinite_map_2EFMERGE A_27a A_27b \in (((ty_2Efinite_map_2Efmap A_27b A_27a)^{(ty_2Efinite_map_2Efmap A_27b A_27a)})^{(ty_2Efinite_map_2Efmap A_27b A_27a)}) \quad (8)$$

Let $c_2Esum_2EISL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Esum_2EISL A_27a A_27b \in (2^{(ty_2Esum_2Esum A_27a A_27b)}) \quad (9)$$

Definition 14 We define $c_2Efinite_map_2EFDOM$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0f \in (ty_2Efinite_map$

Definition 15 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a. (\lambda V1f \in (2^{A_27a}). (ap V1f V0x)))$

Definition 16 We define $c_2Epred_set_2EEMPTY$ to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a. c_2Ebool_2EF)$.

Definition 17 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c_2Ebool_2E_21 2) (\lambda V2t \in$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (10)$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b}})^{A_27a}) \quad (11)$$

Definition 18 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap\ (c_2Epair_2EABS_prod\ A_27a\ A_27b)\ (V0x\ V1y))$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epred_set_2EGSPEC\ A_27a\ A_27b \in ((2^{A_27a})^{(ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b}}) \quad (12)$$

Definition 19 We define $c_2Epred_set_2EUNION$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap\ (c_2Epred_set_2EGSPEC\ A_27a\ A_27a)\ (V0s\ V1t))$

Assume the following.

$$True \quad (13)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (14)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \quad (15)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (16)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \quad (17)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))) \quad (18)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (19)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0x \in A.27a.((V0x = V0x) \Leftrightarrow True)) \quad (20)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (21)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (22)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0t1 \in A.27a.(\forall V1t2 \in A.27a.(((ap (ap (ap (c.2Ebool.2ECOND A.27a) c.2Ebool.2ET) V0t1) V1t2) = V0t1) \wedge ((ap (ap (ap (c.2Ebool.2ECOND A.27a) c.2Ebool.2EF) V0t1) V1t2) = V1t2)))) \quad (23)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3))))) \quad (24)$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x.27 \in 2.(\forall V2y \in 2.(\forall V3y.27 \in 2.(((p V0x) \Leftrightarrow (p V1x.27)) \wedge ((p V1x.27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y.27)))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x.27) \Rightarrow (p V3y.27))))) \quad (25)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in 2.(\forall V2x \in A.27a.(\forall V3x.27 \in A.27a.(\forall V4y \in A.27a.(\forall V5y.27 \in A.27a.(((p V0P) \Leftrightarrow (p V1Q)) \wedge (((p V1Q) \Rightarrow (V2x = V3x.27)) \wedge ((\neg(p V1Q)) \Rightarrow (V4y = V5y.27)))) \Rightarrow ((ap (ap (ap (c.2Ebool.2ECOND A.27a) V0P) V2x) V4y) = (ap (ap (ap (c.2Ebool.2ECOND A.27a) V1Q) V3x.27) V5y.27)))))) \quad (26)$$

Assume the following.

$$\begin{aligned} & \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow (\\ & (ap\ (c_2Efinite_map_2EFDOM\ A_{.27a}\ A_{.27b})\ (c_2Efinite_map_2EFEMPTY \\ & \quad A_{.27a}\ A_{.27b})) = (c_2Epred_set_2EEMPTY\ A_{.27a})) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned} & \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow (\\ & \forall V0f \in (ty_2Efinite_map_2Efmap\ A_{.27a}\ A_{.27b}).(\forall V1g \in \\ & (ty_2Efinite_map_2Efmap\ A_{.27a}\ A_{.27b}).(((ap\ (c_2Efinite_map_2EFDOM \\ & \quad A_{.27a}\ A_{.27b})\ V0f) = (ap\ (c_2Efinite_map_2EFDOM\ A_{.27a}\ A_{.27b})\ V1g)) \wedge \\ & \quad (\forall V2x \in A_{.27a}.((p\ (ap\ (ap\ (c_2Ebool_2EIN\ A_{.27a})\ V2x)\ (ap\ (\\ & \quad c_2Efinite_map_2EFDOM\ A_{.27a}\ A_{.27b})\ V0f)))) \Rightarrow ((ap\ (ap\ (c_2Efinite_map_2EFAPPLY \\ & \quad A_{.27a}\ A_{.27b})\ V0f)\ V2x) = (ap\ (ap\ (c_2Efinite_map_2EFAPPLY\ A_{.27a} \\ & \quad A_{.27b})\ V1g)\ V2x)))))) \Leftrightarrow (V0f = V1g)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow (\\ & \forall V0m \in ((A_{.27a}^{A_{.27a}})^{A_{.27a}}).(\forall V1f \in (ty_2Efinite_map_2Efmap \\ & \quad A_{.27b}\ A_{.27a}).(\forall V2g \in (ty_2Efinite_map_2Efmap\ A_{.27b}\ A_{.27a}). \\ & (((ap\ (c_2Efinite_map_2EFDOM\ A_{.27b}\ A_{.27a})\ (ap\ (ap\ (ap\ (c_2Efinite_map_2EFMERGE \\ & \quad A_{.27a}\ A_{.27b})\ V0m)\ V1f)\ V2g)) = (ap\ (ap\ (c_2Epred_set_2EUNION\ A_{.27b}) \\ & \quad (ap\ (c_2Efinite_map_2EFDOM\ A_{.27b}\ A_{.27a})\ V2g))) \wedge (\forall V3x \in A_{.27b}.((ap\ (ap\ (c_2Efinite_map_2EFAPPLY \\ & \quad A_{.27b}\ A_{.27a})\ (ap\ (ap\ (c_2Efinite_map_2EFMERGE\ A_{.27a}\ A_{.27b}) \\ & \quad V0m)\ V1f)\ V2g))\ V3x) = (ap\ (ap\ (c_2Ebool_2ECOND\ A_{.27a})\ (ap\ c_2Ebool_2E_7E \\ & \quad (ap\ (ap\ (c_2Ebool_2EIN\ A_{.27b})\ V3x)\ (ap\ (c_2Efinite_map_2EFDOM \\ & \quad A_{.27b}\ A_{.27a})\ V1f))))))\ (ap\ (ap\ (c_2Efinite_map_2EFAPPLY\ A_{.27b}\ A_{.27a}) \\ & \quad V2g)\ V3x))\ (ap\ (ap\ (c_2Ebool_2ECOND\ A_{.27a})\ (ap\ c_2Ebool_2E_7E \\ & \quad (ap\ (ap\ (c_2Ebool_2EIN\ A_{.27b})\ V3x)\ (ap\ (c_2Efinite_map_2EFDOM \\ & \quad A_{.27b}\ A_{.27a})\ V2g))))))\ (ap\ (ap\ (c_2Efinite_map_2EFAPPLY\ A_{.27b}\ A_{.27a}) \\ & \quad V1f)\ V3x))\ (ap\ (ap\ V0m\ (ap\ (ap\ (c_2Efinite_map_2EFAPPLY\ A_{.27b}\ A_{.27a}) \\ & \quad V1f)\ V3x))\ (ap\ (ap\ (c_2Efinite_map_2EFAPPLY\ A_{.27b}\ A_{.27a})\ V2g) \\ & \quad V3x)))))))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow (\forall V0x \in A_{.27a}.(\neg(p\ (ap\ (ap \\ & \quad (c_2Ebool_2EIN\ A_{.27a})\ V0x)\ (c_2Epred_set_2EEMPTY\ A_{.27a})))))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow ((\forall V0s \in (2^{A_{.27a}}).((ap\ (\\ & \quad ap\ (c_2Epred_set_2EUNION\ A_{.27a})\ (c_2Epred_set_2EEMPTY\ A_{.27a})) \\ & \quad V0s) = V0s)) \wedge (\forall V1s \in (2^{A_{.27a}}).((ap\ (ap\ (c_2Epred_set_2EUNION \\ & \quad A_{.27a})\ V1s)\ (c_2Epred_set_2EEMPTY\ A_{.27a})) = V1s)))) \end{aligned} \quad (31)$$

Theorem 1

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \quad \forall V0m \in ((A_27b^{A_27b})^{A_27b}). (\forall V1f \in (ty_2Efinite_map_2E fmap \\ & \quad A_27a\ A_27b). (((ap\ (ap\ (ap\ (c_2Efinite_map_2EFMERGE\ A_27b\ A_27a) \\ & \quad V0m)\ V1f)\ (c_2Efinite_map_2EFEMPTY\ A_27a\ A_27b)) = V1f) \wedge ((ap \\ & \quad (ap\ (ap\ (c_2Efinite_map_2EFMERGE\ A_27b\ A_27a)\ V0m)\ (c_2Efinite_map_2EFEMPTY \\ & \quad A_27a\ A_27b))\ V1f) = V1f)))) \end{aligned}$$