

thm_2Efmaptree_2Eapply__path__SNOC (TM- FcFeChGRPbfQCJENMnp7ANiPguMm4eDr2)

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Definition 1 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 2 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 3 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 4 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 5 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t)))$

Definition 6 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 7 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \tag{1}$$

Definition 8 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p$ of type $\iota \Rightarrow \iota$.

Definition 9 We define c_2Eone_2Eone to be $(ap (c_2Emin_2E_40 ty_2Eone_2Eone) (\lambda V0x \in ty_2Eone_2Eone$

Definition 10 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t)))$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty (ty_2Esum_2Esum\ A0\ A1) \tag{2}$$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \tag{3}$$

Definition 11 We define c_Esum_2EINR to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0e \in A_27b. (ap (c_Esum_2EABS$
Let $ty_2Eoption_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow nonempty (ty_2Eoption_2Eoption A0) \quad (4)$$

Let $c_2Eoption_2Eoption_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a. nonempty A_27a \Rightarrow c_2Eoption_2Eoption_ABS A_27a \in ((ty_2Eoption_2Eoption A_27a)^{(ty_2Esum_2Esum A_27a ty_2Eone_2Eone)}) \quad (5)$$

Definition 12 We define $c_2Eoption_2ENONE$ to be $\lambda A_27a : \iota. (ap (c_2Eoption_2Eoption_ABS A_27a) ($
Let $ty_2Efinite_map_2E fmap : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow \forall A1. nonempty A1 \Rightarrow nonempty (ty_2Efinite_map_2E fmap A0 A1) \quad (6)$$

Let $c_2Efinite_map_2E fmap_REP : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a. nonempty A_27a \Rightarrow \forall A_27b. nonempty A_27b \Rightarrow c_2Efinite_map_2E fmap_REP A_27a A_27b \in (((ty_2Esum_2Esum A_27b ty_2Eone_2Eone)^{A_27a})^{(ty_2Efinite_map_2E fmap A_27a A_27b)}) \quad (7)$$

Let $c_2Esum_2EOUTL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a. nonempty A_27a \Rightarrow \forall A_27b. nonempty A_27b \Rightarrow c_2Esum_2EOUTL A_27a A_27b \in (A_27a)^{(ty_2Esum_2Esum A_27a A_27b)} \quad (8)$$

Definition 13 We define $c_2Efinite_map_2EFAPPLY$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0f \in (ty_2Efinite_map$

Definition 14 We define c_2Esum_2EINL to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0e \in A_27a. (ap (c_2Esum_2EABS$

Definition 15 We define $c_2Eoption_2ESOME$ to be $\lambda A_27a : \iota. \lambda V0x \in A_27a. (ap (c_2Eoption_2Eoption_$

Let $c_2Esum_2EISL : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a. nonempty A_27a \Rightarrow \forall A_27b. nonempty A_27b \Rightarrow c_2Esum_2EISL A_27a A_27b \in (2^{(ty_2Esum_2Esum A_27a A_27b)}) \quad (9)$$

Definition 16 We define $c_2Efinite_map_2EFDOM$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0f \in (ty_2Efinite_map$

Definition 17 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a. (\lambda V1f \in (2^{A_27a}). (ap V1f V0x)))$

Definition 18 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A_27a. (\lambda V2t2 \in A_27a. ($

Definition 19 We define $c_2Efinite_map_2EFLOOKUP$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0f \in (ty_2Efinite_map$

Let $ty_2Efmaptree_2Efmaptree : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Efmaptree_2Efmaptree\ A0\ A1) \quad (10)$$

Let $c_2Efmaptree_2Emap : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efmaptree_2Emap\ A_27a\ A_27b \in ((ty_2Efinite_map_2Efmmap\ A_27a\ (ty_2Efmaptree_2Efmaptree\ A_27a\ A_27b))^{(ty_2Efmaptree_2Efmaptree\ A_27a\ A_27b)}) \quad (11)$$

Let $ty_2Elist_2Elist : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Elist_2Elist\ A0) \quad (12)$$

Let $c_2Efmaptree_2Eapply_path : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Efmaptree_2Eapply_path\ A_27a\ A_27b \in (((ty_2Eoption_2Eoption\ (ty_2Efmaptree_2Efmaptree\ A_27a\ A_27b))^{(ty_2Efmaptree_2Efmaptree\ A_27a\ A_27b)})^{(ty_2Elist_2Elist\ A_27a)}) \quad (13)$$

Let $c_2Elist_2EAPPEND : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Elist_2EAPPEND\ A_27a \in (((ty_2Elist_2Elist\ A_27a)^{(ty_2Elist_2Elist\ A_27a)})^{(ty_2Elist_2Elist\ A_27a)}) \quad (14)$$

Let $c_2Elist_2ECONS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Elist_2ECONS\ A_27a \in (((ty_2Elist_2Elist\ A_27a)^{(ty_2Elist_2Elist\ A_27a)})^{A_27a}) \quad (15)$$

Let $c_2Elist_2ENIL : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Elist_2ENIL\ A_27a \in (ty_2Elist_2Elist\ A_27a) \quad (16)$$

Let $c_2Eoption_2Eoption_CASE : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Eoption_2Eoption_CASE\ A_27a\ A_27b \in (((A_27b^{(A_27b^{A_27a})})^{A_27b})^{(ty_2Eoption_2Eoption\ A_27a)}) \quad (17)$$

Assume the following.

$$True \quad (18)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (19)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (20)$$

Assume the following.

$$(\forall V0t \in 2. ((p V0t) \vee (\neg(p V0t)))) \quad (21)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (22)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (23)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (24)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0t1 \in A_27a. (\forall V1t2 \in A_27a. (((ap (ap (ap (c_2Ebool_2ECOND A_27a) c_2Ebool_2ET) V0t1) V1t2) = V0t1) \wedge ((ap (ap (ap (c_2Ebool_2ECOND A_27a) c_2Ebool_2EF) V0t1) V1t2) = V1t2)))) \quad (25)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (26)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0P \in 2. (\forall V1Q \in 2. (\forall V2x \in A_27a. (\forall V3x_27 \in A_27a. (\forall V4y \in A_27a. (\forall V5y_27 \in A_27a. (((p V0P) \Leftrightarrow (p V1Q)) \wedge (((p V1Q) \Rightarrow (V2x = V3x_27)) \wedge ((\neg(p V1Q)) \Rightarrow (V4y = V5y_27)))) \Rightarrow ((ap (ap (ap (c_2Ebool_2ECOND A_27a) V0P) V2x) V4y) = (ap (ap (ap (c_2Ebool_2ECOND A_27a) V1Q) V3x_27) V5y_27)))))))))) \quad (27)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow ((\forall V0t1 \in A_27a. (\forall V1t2 \in A_27a. ((ap (ap (ap (c_2Ebool_2ECOND A_27a) c_2Ebool_2ET) V0t1) V1t2) = V0t1))) \wedge (\forall V2t1 \in A_27a. (\forall V3t2 \in A_27a. ((ap (ap (ap (c_2Ebool_2ECOND A_27a) c_2Ebool_2EF) V2t1) V3t2) = V3t2)))))) \quad (28)$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& (\forall V0ft \in (ty_2Efmptree_2Efmptree\ A_27a\ A_27b).((ap\ (\\
& ap\ (c_2Efmptree_2Eapply_path\ A_27a\ A_27b)\ (c_2Elist_2ENIL \\
& A_27a))\ V0ft) = (ap\ (c_2Eoption_2ESOME\ (ty_2Efmptree_2Efmptree \\
& A_27a\ A_27b))\ V0ft))) \wedge (\forall V1h \in A_27a.(\forall V2t \in (ty_2Elist_2Elist \\
& A_27a).(\forall V3ft \in (ty_2Efmptree_2Efmptree\ A_27a\ A_27b). \\
& ((ap\ (ap\ (c_2Efmptree_2Eapply_path\ A_27a\ A_27b)\ (ap\ (ap\ (c_2Elist_2ECONS \\
& A_27a)\ V1h)\ V2t))\ V3ft) = (ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ (ty_2Eoption_2Eoption \\
& (ty_2Efmptree_2Efmptree\ A_27a\ A_27b)))\ (ap\ (ap\ (c_2Ebool_2EIN \\
& A_27a)\ V1h)\ (ap\ (c_2Efinite_map_2EFDOM\ A_27a\ (ty_2Efmptree_2Efmptree \\
& A_27a\ A_27b))\ (ap\ (c_2Efmptree_2Emap\ A_27a\ A_27b)\ V3ft))))))\ (ap \\
& (ap\ (c_2Efmptree_2Eapply_path\ A_27a\ A_27b)\ V2t)\ (ap\ (ap\ (c_2Efinite_map_2EFAPPLY \\
& A_27a\ (ty_2Efmptree_2Efmptree\ A_27a\ A_27b))\ (ap\ (c_2Efmptree_2Emap \\
& A_27a\ A_27b)\ V3ft))\ V1h)))\ (c_2Eoption_2ENONE\ (ty_2Efmptree_2Efmptree \\
& A_27a\ A_27b))))))))) \\
& \tag{29}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow ((\forall V0l \in (ty_2Elist_2Elist \\
& A_27a).((ap\ (ap\ (c_2Elist_2EAPPEND\ A_27a)\ (c_2Elist_2ENIL\ A_27a)) \\
& V0l) = V0l)) \wedge (\forall V1l1 \in (ty_2Elist_2Elist\ A_27a).(\forall V2l2 \in \\
& (ty_2Elist_2Elist\ A_27a).(\forall V3h \in A_27a.((ap\ (ap\ (c_2Elist_2EAPPEND \\
& A_27a)\ (ap\ (ap\ (c_2Elist_2ECONS\ A_27a)\ V3h)\ V1l1))\ V2l2) = (ap\ (ap \\
& (c_2Elist_2ECONS\ A_27a)\ V3h)\ (ap\ (ap\ (c_2Elist_2EAPPEND\ A_27a) \\
& V1l1)\ V2l2))))))))) \\
& \tag{30}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0P \in (2^{(ty_2Elist_2Elist\ A_27a)}). \\
& (((p\ (ap\ V0P\ (c_2Elist_2ENIL\ A_27a))) \wedge (\forall V1t \in (ty_2Elist_2Elist \\
& A_27a).((p\ (ap\ V0P\ V1t)) \Rightarrow (\forall V2h \in A_27a.(p\ (ap\ V0P\ (ap\ (ap\ (\\
& c_2Elist_2ECONS\ A_27a)\ V2h)\ V1t)))))) \Rightarrow (\forall V3l \in (ty_2Elist_2Elist \\
& A_27a).(p\ (ap\ V0P\ V3l)))))) \\
& \tag{31}
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& (\forall V0v \in A_27b.(\forall V1f \in (A_27b^{A_27a}).((ap\ (ap\ (ap\ (c_2Eoption_2Eoption_CASE \\
& A_27a\ A_27b)\ (c_2Eoption_2ENONE\ A_27a))\ V0v)\ V1f) = V0v))) \wedge (\forall V2x \in \\
& A_27a.(\forall V3v \in A_27b.(\forall V4f \in (A_27b^{A_27a}).((ap\ (ap \\
& (ap\ (c_2Eoption_2Eoption_CASE\ A_27a\ A_27b)\ (ap\ (c_2Eoption_2ESOME \\
& A_27a)\ V2x))\ V3v)\ V4f) = (ap\ V4f\ V2x)))))) \\
& \tag{32}
\end{aligned}$$

Theorem 1

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \quad \forall V0ft \in (ty_2Efmaptree_2Efmaptree\ A_27a\ A_27b).(\forall V1x \in \\ & \quad \quad A_27a.(\forall V2p \in (ty_2Elist_2Elist\ A_27a).((ap\ (ap\ (c_2Efmaptree_2Eapply_path \\ & \quad \quad \quad A_27a\ A_27b)\ (ap\ (ap\ (c_2Elist_2EAPPEND\ A_27a)\ V2p)\ (ap\ (ap\ (c_2Elist_2ECONS \\ & \quad \quad \quad \quad A_27a)\ V1x)\ (c_2Elist_2ENIL\ A_27a))))))\ V0ft) = (ap\ (ap\ (ap\ (c_2Eoption_2Eoption_CASE \\ & \quad \quad \quad \quad (ty_2Efmaptree_2Efmaptree\ A_27a\ A_27b)\ (ty_2Eoption_2Eoption \\ & \quad \quad \quad \quad (ty_2Efmaptree_2Efmaptree\ A_27a\ A_27b)))\ (ap\ (ap\ (c_2Efmaptree_2Eapply_path \\ & \quad \quad \quad \quad \quad A_27a\ A_27b)\ V2p)\ V0ft))\ (c_2Eoption_2ENONE\ (ty_2Efmaptree_2Efmaptree \\ & \quad \quad \quad \quad \quad \quad A_27a\ A_27b)))\ (\lambda V3ft_27 \in (ty_2Efmaptree_2Efmaptree\ A_27a \\ & \quad \quad \quad \quad \quad \quad \quad A_27b).(ap\ (ap\ (c_2Efinite_map_2EFLOOKUP\ A_27a\ (ty_2Efmaptree_2Efmaptree \\ & \quad \quad \quad \quad \quad \quad \quad \quad A_27a\ A_27b))\ (ap\ (c_2Efmaptree_2Emap\ A_27a\ A_27b)\ V3ft_27)\ V1x)))))) \end{aligned}$$