thm_2Efrac_2EFRAC__MUL__COMM (TMVLmca2b33JBgPSpWPrwEfmtu7d8VJBP9f)

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Let $ty\_2Einteger\_2Eint : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \tag{1}$$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{2}$$

Let $ty\_2Efrac\_2Efrac : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Efrac\_2Efrac \tag{3}$$

Let $c\_2Efrac\_2Erep\_\_frac : \iota$ be given. Assume the following.

$$c\_2Efrac\_2Erep\_\_frac \in ((ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)^{ty\_2Efrac\_2Efrac}) \tag{4}$$

Let $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{5}$$

Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 3 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 4 *We define* c_2Efrac_2Efrac__dnm *to be* $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2ESND\ ty$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{6}$$

Definition 5 *We define* c_2Efrac_2Efrac__nmr *to be* $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap\ (c\_2Epair\_2EFST\ ty\_$

Definition 6 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 7 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \tag{7}$$

Definition 8 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2E$

Let $c\_2Efrac\_2Eabs\_\_frac : \iota$ be given. Assume the following.

$$c\_2Efrac\_2Eabs\_\_frac \in (ty\_2Efrac\_2Efrac^{(ty\_2Epair\_2Eprod\ ty\_2Einteger\_2Eint\ ty\_2Einteger\_2Eint)}) \tag{8}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{9}$$

Let $c\_2Einteger\_2Eint\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2} \tag{10}$$

Definition 9 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 10 *We define* c_2Einteger_2Eint__REP *to be* $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (\mathsf{c\_2Emin\_2E\_40}\ (t$

Let $c\_2Einteger\_2Etint\_\_mul : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty} \tag{11}$$

Let $c\_2Einteger\_2Etint\_\_eq : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum} \tag{12}$$

Let $c\_2Einteger\_2Eint\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_ABS\_\_CLASS \in (ty\_2Einteger\_2Eint^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}} \tag{13}$$

Definition 11 *We define* c_2Einteger_2Eint__ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_$

Definition 12 *We define* c_2Einteger_2Eint__mul *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Intege$

Definition 13 *We define* c_2Efrac_2Efrac__mul *to be* $\lambda V0f1 \in ty\_2Efrac\_2Efrac.\lambda V1f2 \in ty\_2Efrac\_2E$

Let $c\_2Einteger\_2Etint\_\_neg : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \tag{14}$$

Definition 14 *We define* c_2Einteger_2Eint__neg *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap$ c_2Einteger_2Eint_

Let $c\_2Einteger\_2Etint\_\_add : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty} \tag{15}$$

Definition 15 *We define* c_2Einteger_2Eint__add *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Eintege$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{16}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{17}$$

Definition 16 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 17 *We define* c_2Earithmetic_2EZERO *to be* c_2Enum_2E0*.*

Let $c\_2Enum\_2EREP\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EREP\_\_num \in (omega^{ty\_2Enum\_2Enum}) \tag{18}$$

Let $c\_2Enum\_2ESUC\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2ESUC\_\_REP \in (omega^{omega}) \tag{19}$$

Definition 18 *We define* c_2Enum_2ESUC *to be* $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_\_num$

Let $c\_2Earithmetic\_2E\_2B : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{20}$$

Definition 19 *We define* c_2Earithmetic_2EBIT1 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Definition 20 *We define* c_2Earithmetic_2ENUMERAL *to be* $\lambda V0x \in ty\_2Enum\_2Enum.V0x$.

Definition 21 *We define* c_2Einteger_2Etint__1 *to be* $(ap\ (ap$ (c_2Epair_2E_2C $ty\_2Enum\_2Enum\ ty\_2Enu$

Definition 22 *We define* c_2Einteger_2Eint__1 *to be* $(ap$ c_2Einteger_2Eint__ABS c_2Einteger_2Etint__1$)$.

Definition 23 *We define* c_2Einteger_2Etint__0 *to be* $(ap\ (ap\ ($c_2Epair_2E_2C $ty\_2Enum\_2Enum\ ty\_2Enu$

Definition 24 *We define* c_2Einteger_2Eint__0 *to be* $(ap$ c_2Einteger_2Eint__ABS c_2Einteger_2Etint__0$)$.

Let $ty\_2Ering\_2Ering : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ering\_2Ering\ A0) \tag{21}$$

Let $c\_2Ering\_2Erecordtype\_2Ering : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ering\_2Erecordtype\_2Ering\ A\_27a \in ((((((ty\_2Ering\_2Ering\ A\_27a)^{(A\_27a^{A\_27a})})^{((A\_27a^{A\_27a})^{A\_27a})})^{((A\_27a^{A\_27a})^{A\_27a})})^{A\_27a})^{A\_27a}) \tag{22}$$

Let $ty\_2Ecanonical\_2Ecanonical\_\_sum : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ecanonical\_2Ecanonical\_\_sum\ A0) \tag{23}$$

Let $ty\_2EringNorm\_2Epolynom : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2EringNorm\_2Epolynom\ A0) \tag{24}$$

Let $c\_2EringNorm\_2Epolynom\_\_normalize : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EringNorm\_2Epolynom\_\_normalize\ A\_27a \in (((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2EringNorm\_2Epolynom\ A\_27a)})^{(ty\_2Ering\_2Ering\ A} \tag{25}$$

Definition 25 *We define* c_2EintegerRing_2Eint__polynom__normalize *to be* $(ap\ (c\_2EringNorm\_2Epolynon$

Let $c\_2Ering\_2Ering\_\_RM : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ering\_2Ering\_\_RM\ A\_27a \in (((A\_27a^{A\_27a})^{A\_27a})^{(ty\_2Ering\_2Ering\ A\_27a)}) \tag{26}$$

Let $c\_2Ering\_2Ering\_\_RP : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ering\_2Ering\_\_RP\ A\_27a \in (((A\_27a^{A\_27a})^{A\_27a})^{(ty\_2Ering\_2Ering\ A\_27a)}) \tag{27}$$

Let $c\_2Ering\_2Ering\_\_R1 : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ering\_2Ering\_\_R1\ A\_27a \in (A\_27a^{(ty\_2Ering\_2Ering\ A\_27a)}) \tag{28}$$

Let $c\_2Ering\_2Ering\_\_R0 : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ering\_2Ering\_\_R0\ A\_27a \in (A\_27a^{(ty\_2Ering\_2Ering\ A\_27a)}) \tag{29}$$

Let $ty\_2Esemi\_\_ring\_2Esemi\_\_ring : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Esemi\_\_ring\_2Esemi\_\_ring\ A0) \quad (30)$$

Let $c\_2Esemi\_\_ring\_2Erecordtype\_2Esemi\_\_ring : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Esemi\_\_ring\_2Erecordtype\_2Esemi\_\_ring\ A\_27a \in (((((ty\_2Esemi\_\_ring\_2Esemi\_\_ring\ A\_27a)^{((A\_27a^{A\_27a})^{A\_27a})})^{((A\_27a^{A\_27a})^{A\_27a})})^{A\_27a})^{A\_27a}) \quad (31)$$

Definition 26 *We define* c_2Ering_2Esemi__ring__of *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A\_27a).(a$

Let $ty\_2Equote\_2Evarmap : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Equote\_2Evarmap\ A0) \quad (32)$$

Let $c\_2Ecanonical\_2Eics\_\_aux : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Eics\_\_aux\ A\_27a \in ((((A\_27a^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{A\_27a})^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{(ty\_2Esemi\_\_ring\_2Esemi\_\_r} \quad (33)$$

Definition 27 *We define* c_2EringNorm_2Er__ics__aux *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A\_27a).($

Definition 28 *We define* c_2EintegerRing_2Eint__r__ics__aux *to be* $(ap$ (c_2EringNorm_2Er__ics__aux ty_2Eint

Let $ty\_2Equote\_2Eindex : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Equote\_2Eindex \quad (34)$$

Let $ty\_2Elist\_2Elist : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Elist\_2Elist\ A0) \quad (35)$$

Let $c\_2Ecanonical\_2Einterp\_\_m : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Einterp\_\_m\ A\_27a \in ((((A\_27a^{(ty\_2Elist\_2Elist\ ty\_2Equote\_2Eindex)})^{A\_27a})^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{(ty\_2Esemi\_\_ring\_2Esemi\_\_ring} \quad (36)$$

Definition 29 *We define* c_2EringNorm_2Er__interp__m *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A\_27a$

Definition 30 *We define* c_2EintegerRing_2Eint__r__interp__m *to be* $(ap$ (c_2EringNorm_2Er__interp__m ty_2E

Let $c\_2Ecanonical\_2Einterp\_\_vl : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Einterp\_\_vl\ A\_27a \in (((A\_27a^{(ty\_2Elist\_2Elist\ ty\_2Equote\_2Eindex)})^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{(ty\_2Esemi\_\_ring\_2Esemi\_\_ring\ A\_27a)}) \quad (37)$$

Definition 31 *We define* c_2EringNorm_2Er__interp__vl *to be* $\lambda A\_27a:\iota.\lambda V0r\in(ty\_2Ering\_2Ering\ A\_27a)$

Definition 32 *We define* c_2EintegerRing_2Eint__r__interp__vl *to be* $(ap$ (c_2EringNorm_2Er__interp__vl ty_2E

Let $c\_2Ecanonical\_2Eivl\_\_aux:\iota\Rightarrow\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a\Rightarrow c\_2Ecanonical\_2Eivl\_\_aux\ A\_27a\in ((((A\_27a^{(ty\_2Elist\_2Elist\ ty\_2Equote\_2Eindex)})^{ty\_2Equote\_2Eindex})^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{(ty\_2Esemi\_\_ring\_2} \quad (38)$$

Definition 33 *We define* c_2EringNorm_2Er__ivl__aux *to be* $\lambda A\_27a:\iota.\lambda V0r\in(ty\_2Ering\_2Ering\ A\_27a).($

Definition 34 *We define* c_2EintegerRing_2Eint__r__ivl__aux *to be* $(ap$ (c_2EringNorm_2Er__ivl__aux ty_2Einte

Let $c\_2Ecanonical\_2Ecanonical\_\_sum\_\_simplify:\iota\Rightarrow\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a\Rightarrow c\_2Ecanonical\_2Ecanonical\_\_sum\_\_simplify\ A\_27a\in(((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Esemi\_\_ri} \quad (39)$$

Definition 35 *We define* c_2EringNorm_2Er__canonical__sum__simplify *to be* $\lambda A\_27a:\iota.\lambda V0r\in(ty\_2Ering\_2$

Definition 36 *We define* c_2EintegerRing_2Eint__r__canonical__sum__simplify *to be* $(ap$ (c_2EringNorm_2Er__canonical__sum__simplify $ty\_2Einteger\_2Eint)$ $(ap\ (ap\ (ap\ (ap\ (ap\ (c\_2Ering\_2E$

Let $c\_2Ecanonical\_2Ecanonical\_\_sum\_\_prod:\iota\Rightarrow\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a\Rightarrow c\_2Ecanonical\_2Ecanonical\_\ A\_27a\in((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Ecanonic} \quad (40)$$

Definition 37 *We define* c_2EringNorm_2Er__canonical__sum__prod *to be* $\lambda A\_27a:\iota.\lambda V0r\in(ty\_2Ering\_2E$

Definition 38 *We define* c_2EintegerRing_2Eint__r__canonical__sum__prod *to be* $(ap$ (c_2EringNorm_2Er__canonical__sum__prod $ty\_2Einteger\_2Eint)$ $(ap\ (ap\ (ap\ (ap\ (ap\ (c\_2Ering\_2Ereco$

Let $c\_2Ecanonical\_2Ecanonical\_\_sum\_\_scalar3:\iota\Rightarrow\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a\Rightarrow c\_2Ecanonical\_2Ecanonical\_\ A\_27a\in(((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Elist\_2E} \quad (41)$$

Definition 39 *We define* c_2EringNorm_2Er__canonical__sum__scalar3 *to be* $\lambda A\_27a:\iota.\lambda V0r\in(ty\_2Ering\_2$

Definition 40 *We define* c_2EintegerRing_2Eint__r__canonical__sum__scalar3 *to be* $(ap$ (c_2EringNorm_2Er__canonical__sum__scalar3 $ty\_2Einteger\_2Eint)$ $(ap\ (ap\ (ap\ (ap\ (ap\ (c\_2Ering\_2Ere$

Let $c\_2Ecanonical\_2Ecanonical\_\_sum\_\_scalar2 : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Ecanonical\_\_sum \ldots$$
$$A\_27a \in ((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Elist\_2El \ldots} \tag{42}$$

Definition 41 *We define* c_2EringNorm_2Er__canonical__sum__scalar2 *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2 \ldots$

Definition 42 *We define* c_2EintegerRing_2Eint__r__canonical__sum__scalar2 *to be* $(ap$ (c_2EringNorm_2Er__canonical__sum__scalar2 $ty\_2Einteger\_2Eint)$ $(ap\ (ap\ (ap\ (ap\ (ap\ (c\_2Ering\_2Erec \ldots$

Let $c\_2Ecanonical\_2Ecanonical\_\_sum\_\_scalar : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Ecanonical\_\_sum\_\_scalar$$
$$A\_27a \in ((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{A\_27a})^{(ty\_2Es \ldots} \tag{43}$$

Definition 43 *We define* c_2EringNorm_2Er__canonical__sum__scalar *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2E \ldots$

Definition 44 *We define* c_2EintegerRing_2Eint__r__canonical__sum__scalar *to be* $(ap$ (c_2EringNorm_2Er__canonical__sum__scalar $ty\_2Einteger\_2Eint)$ $(ap\ (ap\ (ap\ (ap\ (ap\ (c\_2Ering\_2Erec \ldots$

Let $c\_2Ecanonical\_2Evarlist\_\_insert : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Evarlist\_\_i \ldots$$
$$A\_27a \in ((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Elist\_2El \ldots} \tag{44}$$

Definition 45 *We define* c_2EringNorm_2Er__varlist__insert *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A\_2 \ldots$

Definition 46 *We define* c_2EintegerRing_2Eint__r__varlist__insert *to be* $(ap$ (c_2EringNorm_2Er__varlist__inser …

Let $c\_2Ecanonical\_2Emonom\_\_insert : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Emonom \ldots$$
$$A\_27a \in (((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Elist\_2E \ldots} \tag{45}$$

Definition 47 *We define* c_2EringNorm_2Er__monom__insert *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A \ldots$

Definition 48 *We define* c_2EintegerRing_2Eint__r__monom__insert *to be* $(ap$ (c_2EringNorm_2Er__monom__in …

Let $c\_2Ecanonical\_2ENil\_\_monom : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2ENil\_\_monom\ A\_27a \in (ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a) \tag{46}$$

Let $c\_2Ecanonical\_2ECons\_\_varlist : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2ECons\_\_varlist\ A\_27a \in (((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Elist\_2Eli} \quad (47)$$

Let $ty\_2EternaryComparisons\_2Eordering : \iota$ be given. Assume the following.

$$nonempty\ ty\_2EternaryComparisons\_2Eordering \quad (48)$$

Let $c\_2EternaryComparisons\_2Elist\_\_compare : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2EternaryComparisons\ A\_27a\ A\_27b \in (((ty\_2EternaryComparisons\_2Eordering^{(ty\_2Elist\_2Elist\ A\_27b)})^{(ty\_2Elist\_2Elist\ A\_27a)})^{((ty} \quad (49)$$

Let $c\_2EternaryComparisons\_2Eordering2num : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2Eordering2num \in (ty\_2Enum\_2Enum^{ty\_2EternaryComparisons\_2Eordering}) \quad (50)$$

Definition 49 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 50 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Definition 51 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2E}$

Definition 52 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (\mathsf{c\_2Emin\_2E\_40}$

Definition 53 *We define* c_2Eprim__rec_2E_3C *to be* $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enu$

Definition 54 *We define* c_2EternaryComparisons_2Eordering__CASE *to be* $\lambda A\_27a : \iota.\lambda V0x \in ty\_2Eternary$

Let $c\_2Ecanonical\_2ECons\_\_monom : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2ECons\_\_monom\ A\_27a \in ((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Elist\_2Elist\ ty\_2Eq} \quad (51)$$

Let $c\_2Ecanonical\_2Ecanonical\_\_sum\_\_merge : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Ecanonical\_\_s\ A\_27a \in ((((ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Ecanonic} \quad (52)$$

Definition 55 *We define* c_2EringNorm_2Er__canonical__sum__merge *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2E$

Definition 56 *We define* c_2EintegerRing_2Eint__r__canonical__sum__merge *to be* $(ap\ (\mathsf{c\_2EringNorm\_2Er\_\_canonical\_\_sum\_\_merge}\ ty\_2Einteger\_2Eint)\ (ap\ (ap\ (ap\ (ap\ (ap\ (c\_2Ering\_2Erec$

Let $c\_2Equote\_2EEmpty\_\_vm : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Equote\_2EEmpty\_\_vm\ A\_27a \in (ty\_2Equote\_2Evarmap\ A\_27a) \tag{53}$$

Let $c\_2Equote\_2ENode\_\_vm : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Equote\_2ENode\_\_vm\ A\_27a \in (((((ty\_2Equote\_2Evarmap\ A\_27a)^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{A\_27a}) \tag{54}$$

Let $c\_2EringNorm\_2EPopp : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EringNorm\_2EPopp\ A\_27a \in ((ty\_2EringNorm\_2Epolynom\ A\_27a)^{(ty\_2EringNorm\_2Epolynom\ A\_27a)}) \tag{55}$$

Let $c\_2EringNorm\_2EPmult : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EringNorm\_2EPmult\ A\_27a \in (((ty\_2EringNorm\_2Epolynom\ A\_27a)^{(ty\_2EringNorm\_2Epolynom\ A\_27a)})^{(ty\_2EringNorm\_2Epolynom\ A\_27a)}) \tag{56}$$

Let $c\_2EringNorm\_2EPplus : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EringNorm\_2EPplus\ A\_27a \in (((ty\_2EringNorm\_2Epolynom\ A\_27a)^{(ty\_2EringNorm\_2Epolynom\ A\_27a)})^{(ty\_2EringNorm\_2Epolynom\ A\_27a)}) \tag{57}$$

Let $c\_2Equote\_2Evarmap\_\_find : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Equote\_2Evarmap\_\_find\ A\_27a \in ((A\_27a^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{ty\_2Equote\_2Eindex}) \tag{58}$$

Let $c\_2EringNorm\_2EPvar : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EringNorm\_2EPvar\ A\_27a \in ((ty\_2EringNorm\_2Epolynom\ A\_27a)^{ty\_2Equote\_2Eindex}) \tag{59}$$

Let $c\_2EringNorm\_2EPconst : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EringNorm\_2EPconst\ A\_27a \in ((ty\_2EringNorm\_2Epolynom\ A\_27a)^{A\_27a}) \tag{60}$$

Definition 57 *We define* c_2EringNorm_2Epolynom__simplify *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A$

Definition 58 *We define* c_2EintegerRing_2Eint__polynom__simplify *to be* $(ap$ (c_2EringNorm_2Epolynom__sim

Let $c\_2Ecanonical\_2Einterp\_\_cs : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ecanonical\_2Einterp\_\_cs\ A\_27a \in (((A\_27a^{(ty\_2Ecanonical\_2Ecanonical\_\_sum\ A\_27a)})^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{(ty\_2Esemi\_\_ring\_2Esemi\_\_ring\ A\_2} \tag{61}$$

Definition 59 *We define* c_2EringNorm_2Er__interp__cs *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A\_27a)$

Definition 60 *We define* c_2EintegerRing_2Eint__r__interp__cs *to be* $(ap\ ($c_2EringNorm_2Er__interp__cs ty_2E

Let $c\_2EringNorm\_2Einterp\_\_p : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EringNorm\_2Einterp\_\_p\ A\_27a \in (((A\_27a^{(ty\_2EringNorm\_2Epolynom\ A\_27a)})^{(ty\_2Equote\_2Evarmap\ A\_27a)})^{(ty\_2Ering\_2Ering\ A\_27a)}) \quad (62)$$

Definition 61 *We define* c_2EintegerRing_2Eint__interp__p *to be* $(ap\ (c\_2EringNorm\_2Einterp\_\_p\ ty\_2Ein$

Let $c\_2Ering\_2Ering\_\_RN : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ering\_2Ering\_\_RN\ A\_27a \in ((A\_27a^{A\_27a})^{(ty\_2Ering\_2Ering\ A\_27a)}) \quad (63)$$

Definition 62 *We define* c_2Ering_2Eis__ring *to be* $\lambda A\_27a : \iota.\lambda V0r \in (ty\_2Ering\_2Ering\ A\_27a).(ap\ (ap$ c

Let $c\_2Equote\_2ERight\_\_idx : \iota$ be given. Assume the following.

$$c\_2Equote\_2ERight\_\_idx \in (ty\_2Equote\_2Eindex^{ty\_2Equote\_2Eindex}) \quad (64)$$

Let $c\_2Equote\_2ELeft\_\_idx : \iota$ be given. Assume the following.

$$c\_2Equote\_2ELeft\_\_idx \in (ty\_2Equote\_2Eindex^{ty\_2Equote\_2Eindex}) \quad (65)$$

Let $c\_2Equote\_2EEnd\_\_idx : \iota$ be given. Assume the following.

$$c\_2Equote\_2EEnd\_\_idx \in ty\_2Equote\_2Eindex \quad (66)$$

Let $c\_2Equote\_2Eindex\_\_compare : \iota$ be given. Assume the following.

$$c\_2Equote\_2Eindex\_\_compare \in ((ty\_2EternaryComparisons\_2Eordering^{ty\_2Equote\_2Eindex})^{ty\_2Equote\_2E} \quad (67)$$

Let $c\_2EternaryComparisons\_2ELESS : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2ELESS \in ty\_2EternaryComparisons\_2Eordering \quad (68)$$

Definition 63 *We define* c_2Equote_2Eindex__lt *to be* $\lambda V0i1 \in ty\_2Equote\_2Eindex.\lambda V1i2 \in ty\_2Equote\_2$

Let $c\_2EternaryComparisons\_2Enum2ordering : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2Enum2ordering \in (ty\_2EternaryComparisons\_2Eordering^{ty\_2Enum\_2Enum} \quad (69)$$

Definition 64 *We define* c_2Earithmetic_2EBIT2 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Let $c\_2EternaryComparisons\_2EGREATER : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2EGREATER \in ty\_2EternaryComparisons\_2Eordering \quad (70)$$

Let $c\_2EternaryComparisons\_2EEQUAL : \iota$ be given. Assume the following.

$$c\_2EternaryComparisons\_2EEQUAL \in ty\_2EternaryComparisons\_2Eordering \quad (71)$$

Let $c\_2Elist\_2ECONS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Elist\_2ECONS\ A\_27a \in (((ty\_2Elist\_2Elist\ A\_27a)^{(ty\_2Elist\_2Elist\ A\_27a)})^{A\_27a}) \quad (72)$$

Let $c\_2Elist\_2ENIL : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Elist\_2ENIL\ A\_27a \in (ty\_2Elist\_2Elist\ A\_27a) \quad (73)$$

Let $c\_2EternaryComparisons\_2Elist\_\_merge : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2EternaryComparisons\_2Elist\_\_merge\ A\_27a \in ((((ty\_2Elist\_2Elist\ A\_27a)^{(ty\_2Elist\_2Elist\ A\_27a)})^{(ty\_2Elist\_2Elist\ A\_27a)})^{((2^{A\_27a})^{A\_27a})}) \quad (74)$$

Assume the following.

$$True \quad (75)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (76)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t1 \in A\_27a.(\forall V1t2 \in A\_27a.(((ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ A\_27a)\ \mathsf{c\_2Ebool\_2ET})\ V0t1)\ V1t2) = V0t1) \wedge ((ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ A\_27a)\ \mathsf{c\_2Ebool\_2EF})\ V0t1)\ V1t2) = V1t2)))) \quad (77)$$

Assume the following.

$$
\begin{array}{c}
\forall A\_27a.nonempty\ A\_27a \Rightarrow ((p\ (ap\ (\mathsf{c\_2Ering\_2Eis\_\_ring}\ ty\_2Einteger\_2Eint) \\
(ap\ (ap\ (ap\ (ap\ (ap\ (c\_2Ering\_2Erecordtype\_2Ering\ ty\_2Einteger\_2Eint) \\
\mathsf{c\_2Einteger\_2Eint\_\_0})\ \mathsf{c\_2Einteger\_2Eint\_\_1})\ \mathsf{c\_2Einteger\_2Eint\_\_add}) \\
\mathsf{c\_2Einteger\_2Eint\_\_mul})\ \mathsf{c\_2Einteger\_2Eint\_\_neg})))\wedge((\forall V0vm \in \\
(ty\_2Equote\_2Evarmap\ ty\_2Einteger\_2Eint).(\forall V1p \in (ty\_2EringNorm\_2Epolynom \\
ty\_2Einteger\_2Eint).((ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p} \\
V0vm)\ V1p) = (ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_r\_\_interp\_\_cs}\ V0vm) \\
(ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_polynom\_\_simplify}\ V1p)))))\wedge(((\\
(\forall V2vm \in (ty\_2Equote\_2Evarmap\ ty\_2Einteger\_2Eint).(\forall V3c \in \\
ty\_2Einteger\_2Eint.((ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p} \\
V2vm)\ (ap\ (c\_2EringNorm\_2EPconst\ ty\_2Einteger\_2Eint)\ V3c)) = \\
V3c)))\wedge((\forall V4vm \in (ty\_2Equote\_2Evarmap\ ty\_2Einteger\_2Eint). \\
(\forall V5i \in ty\_2Equote\_2Eindex.((ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p} \\
V4vm)\ (ap\ (c\_2EringNorm\_2EPvar\ ty\_2Einteger\_2Eint)\ V5i)) = (ap \\
(ap\ (c\_2Equote\_2Evarmap\_\_find\ ty\_2Einteger\_2Eint)\ V5i)\ V4vm))))\wedge \\
((\forall V6vm \in (ty\_2Equote\_2Evarmap\ ty\_2Einteger\_2Eint).(\\
\forall V7p1 \in (ty\_2EringNorm\_2Epolynom\ ty\_2Einteger\_2Eint). \\
(\forall V8p2 \in (ty\_2EringNorm\_2Epolynom\ ty\_2Einteger\_2Eint). \\
((ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p}\ V6vm)\ (ap\ (ap\ (c\_2EringNorm\_2EPplus \\
ty\_2Einteger\_2Eint)\ V7p1)\ V8p2)) = (ap\ (ap\ \mathsf{c\_2Einteger\_2Eint\_\_add} \\
(ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p}\ V6vm)\ V7p1))\ (ap\ (ap \\
\mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p}\ V6vm)\ V8p2))))))\wedge((\forall V9vm \in \\
(ty\_2Equote\_2Evarmap\ ty\_2Einteger\_2Eint).(\forall V10p1 \in (\\
ty\_2EringNorm\_2Epolynom\ ty\_2Einteger\_2Eint).(\forall V11p2 \in \\
(ty\_2EringNorm\_2Epolynom\ ty\_2Einteger\_2Eint).((ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p} \\
V9vm)\ (ap\ (ap\ (c\_2EringNorm\_2EPmult\ ty\_2Einteger\_2Eint)\ V10p1) \\
V11p2)) = (ap\ (ap\ \mathsf{c\_2Einteger\_2Eint\_\_mul}\ (ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p} \\
V9vm)\ V10p1))\ (ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p}\ V9vm) \\
V11p2))))))\wedge(\forall V12vm \in (ty\_2Equote\_2Evarmap\ ty\_2Einteger\_2Eint). \\
(\forall V13p1 \in (ty\_2EringNorm\_2Epolynom\ ty\_2Einteger\_2Eint). \\
((ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p}\ V12vm)\ (ap\ (c\_2EringNorm\_2EPopp \\
ty\_2Einteger\_2Eint)\ V13p1)) = (ap\ \mathsf{c\_2Einteger\_2Eint\_\_neg}\ (ap \\
(ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_interp\_\_p}\ V12vm)\ V13p1)))))))))\wedge \\
((\forall V14x \in A\_27a.(\forall V15v2 \in (ty\_2Equote\_2Evarmap\ A\_27a). \\
(\forall V16v1 \in (ty\_2Equote\_2Evarmap\ A\_27a).((ap\ (ap\ (c\_2Equote\_2Evarmap\_\_find \\
A\_27a)\ c\_2Equote\_2EEnd\_\_idx)\ (ap\ (ap\ (ap\ (c\_2Equote\_2ENode\_\_vm \\
A\_27a)\ V14x)\ V16v1)\ V15v2)) = V14x))))\wedge((\forall V17x \in A\_27a.(\\
\forall V18v2 \in (ty\_2Equote\_2Evarmap\ A\_27a).(\forall V19v1 \in (\\
ty\_2Equote\_2Evarmap\ A\_27a).(\forall V20i1 \in ty\_2Equote\_2Eindex. \\
((ap\ (ap\ (c\_2Equote\_2Evarmap\_\_find\ A\_27a)\ (ap\ c\_2Equote\_2ERight\_\_idx \\
V20i1))\ (ap\ (ap\ (ap\ (c\_2Equote\_2ENode\_\_vm\ A\_27a)\ V17x)\ V19v1)\ V18v2)) = \\
(ap\ (ap\ (c\_2Equote\_2Evarmap\_\_find\ A\_27a)\ V20i1)\ V18v2))))))\wedge \\
((\forall V21x \in A\_27a.(\forall V22v2 \in (ty\_2Equote\_2Evarmap\ A\_27a). \\
(\forall V23v1 \in (ty\_2Equote\_2Evarmap\ A\_27a).(\forall V24i1 \in \\
ty\_2Equote\_2Eindex.((ap\ (ap\ (c\_2Equote\_2Evarmap\_\_find\ A\_27a) \\
(ap\ c\_2Equote\_2ELeft\_\_idx\ V24i1))\ (ap\ (ap\ (ap\ (c\_2Equote\_2ENode\_\_vm \\
A\_27a)\ V21x)\ V23v1)\ V22v2)) = (ap\ (ap\ (c\_2Equote\_2Evarmap\_\_find \\
A\_27a)\ V24i1)\ V23v1))))))\wedge(\forall V25i \in ty\_2Equote\_2Eindex. \\
((ap\ (ap\ (c\_2Equote\_2Evarmap\_\_find\ A\_27a)\ V25i)\ (c\_2Equote\_2EEmpty\_\_vm \\
A\_27a)) = (ap\ (\mathsf{c\_2Emin\_2E\_40}\ A\_27a)\ (\lambda V26x \in A\_27a.\mathsf{c\_2Ebool\_2ET}))))))))\wedge \\
(((\forall V27t2 \in (ty\_2Ecanonical\_2Ecanonical\_\_sum\ ty\_2Einteger\_2Eint). \\
(\forall V28t1 \in (ty\_2Ecanonical\_2Ecanonical\_\_sum\ ty\_2Einteger\_2Eint). \\
(\forall V29l2 \in (ty\_2Elist\_2Elist\ ty\_2Equote\_2Eindex).(\forall V30l1 \in \\
(ty\_2Elist\_2Elist\ ty\_2Equote\_2Eindex).(\forall V31c2 \in ty\_2Einteger\_2Eint. \\
(\forall V32c1 \in ty\_2Einteger\_2Eint.((ap\ (ap\ \mathsf{c\_2EintegerRing\_2Eint\_\_r\_\_canonical\_\_sum\_\_merge} \\
(ap\ (ap\ (ap\ (c\_2Ecanonical\_2ECons\_\_monom\ ty\_2Einteger\_2Eint) \\
V32c1)\ V30l1)\ V28t1))\ (ap\ (ap\ (ap\ (c\_2Ecanonical\_2ECons\_\_monom \\
ty\_2Einteger\_2Eint)\ V31c2)\ V29l2)\ V27t2)) = (ap\ (ap\ (ap\ (ap\ (\mathsf{c\_2EternaryComparisons\_2Eordering\_\_CA} \\
(ty\_2Ecanonical\_2Ecanonical\_\_sum\ ty\_2Einteger\_2Eint))\ (ap
\end{array}
$$

Assume the following.

$$
\begin{array}{c}
(((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare\ c\_2Equote\_2EEnd\_\_idx) \\
c\_2Equote\_2EEnd\_\_idx) = c\_2EternaryComparisons\_2EEQUAL) \wedge (\\
(\forall V0v10 \in ty\_2Equote\_2Eindex.((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare \\
c\_2Equote\_2EEnd\_\_idx)\ (ap\ c\_2Equote\_2ELeft\_\_idx\ V0v10)) = c\_2EternaryComparisons\_2ELESS) \\
((\forall V1v11 \in ty\_2Equote\_2Eindex.((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare \\
c\_2Equote\_2EEnd\_\_idx)\ (ap\ c\_2Equote\_2ERight\_\_idx\ V1v11)) = c\_2EternaryComparisons\_2ELESS \\
((\forall V2v2 \in ty\_2Equote\_2Eindex.((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare \\
(ap\ c\_2Equote\_2ELeft\_\_idx\ V2v2))\ c\_2Equote\_2EEnd\_\_idx) = c\_2EternaryComparisons\_2EGREATE \\
((\forall V3v3 \in ty\_2Equote\_2Eindex.((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare \\
(ap\ c\_2Equote\_2ERight\_\_idx\ V3v3))\ c\_2Equote\_2EEnd\_\_idx) = c\_2EternaryComparisons\_2EGREATE \\
((\forall V4n\_27 \in ty\_2Equote\_2Eindex.(\forall V5m\_27 \in ty\_2Equote\_2Eindex. \\
((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare\ (ap\ c\_2Equote\_2ELeft\_\_idx \\
V4n\_27))\ (ap\ c\_2Equote\_2ELeft\_\_idx\ V5m\_27)) = (ap\ (ap\ c\_2Equote\_2Eindex\_\_compare \\
V4n\_27)\ V5m\_27)))) \wedge ((\forall V6n\_27 \in ty\_2Equote\_2Eindex.(\forall V7m\_27 \in \\
ty\_2Equote\_2Eindex.((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare\ (ap \\
c\_2Equote\_2ELeft\_\_idx\ V6n\_27))\ (ap\ c\_2Equote\_2ERight\_\_idx\ V7m\_27)) = \\
c\_2EternaryComparisons\_2ELESS))) \wedge ((\forall V8n\_27 \in ty\_2Equote\_2Eindex. \\
(\forall V9m\_27 \in ty\_2Equote\_2Eindex.((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare \\
(ap\ c\_2Equote\_2ERight\_\_idx\ V8n\_27))\ (ap\ c\_2Equote\_2ERight\_\_idx \\
V9m\_27)) = (ap\ (ap\ c\_2Equote\_2Eindex\_\_compare\ V8n\_27)\ V9m\_27)))) \wedge \\
(\forall V10n\_27 \in ty\_2Equote\_2Eindex.(\forall V11m\_27 \in ty\_2Equote\_2Eindex. \\
((ap\ (ap\ c\_2Equote\_2Eindex\_\_compare\ (ap\ c\_2Equote\_2ERight\_\_idx \\
V10n\_27))\ (ap\ c\_2Equote\_2ELeft\_\_idx\ V11m\_27)) = c\_2EternaryComparisons\_2EGREATER)))))))
\end{array}
\tag{79}
$$

Assume the following.

$$
\begin{array}{c}
((\forall V0x \in ty\_2EternaryComparisons\_2Eordering.((V0x = V0x) \Leftrightarrow \\
True)) \wedge (((c\_2EternaryComparisons\_2ELESS = c\_2EternaryComparisons\_2EEQUAL) \Leftrightarrow \\
False) \wedge (((c\_2EternaryComparisons\_2ELESS = c\_2EternaryComparisons\_2EGREATER) \Leftrightarrow \\
False) \wedge (((c\_2EternaryComparisons\_2EEQUAL = c\_2EternaryComparisons\_2EGREATER) \Leftrightarrow \\
False) \wedge (((c\_2EternaryComparisons\_2EEQUAL = c\_2EternaryComparisons\_2ELESS) \Leftrightarrow \\
False) \wedge (((c\_2EternaryComparisons\_2EGREATER = c\_2EternaryComparisons\_2ELESS) \Leftrightarrow \\
False) \wedge (((c\_2EternaryComparisons\_2EGREATER = c\_2EternaryComparisons\_2EEQUAL) \Leftrightarrow \\
False) \wedge ((((ap\ c\_2EternaryComparisons\_2Eordering2num\ c\_2EternaryComparisons\_2ELESS) = \\
\mathsf{c\_2Enum\_2E0}) \wedge (((ap\ c\_2EternaryComparisons\_2Eordering2num \\
c\_2EternaryComparisons\_2EEQUAL) = (ap\ \mathsf{c\_2Earithmetic\_2ENUMERAL} \\
(ap\ \mathsf{c\_2Earithmetic\_2EBIT1}\ \mathsf{c\_2Earithmetic\_2EZERO}))) \wedge ((ap\ c\_2EternaryComparisons\_2Eordering2num \\
c\_2EternaryComparisons\_2EGREATER) = (ap\ \mathsf{c\_2Earithmetic\_2ENUMERAL} \\
(ap\ \mathsf{c\_2Earithmetic\_2EBIT2}\ \mathsf{c\_2Earithmetic\_2EZERO}))))) \wedge (((ap \\
c\_2EternaryComparisons\_2Enum2ordering\ \mathsf{c\_2Enum\_2E0}) = c\_2EternaryComparisons\_2ELESS) \wedge \\
(((ap\ c\_2EternaryComparisons\_2Enum2ordering\ (ap\ \mathsf{c\_2Earithmetic\_2ENUMERAL} \\
(ap\ \mathsf{c\_2Earithmetic\_2EBIT1}\ \mathsf{c\_2Earithmetic\_2EZERO}))) = c\_2EternaryComparisons\_2EEQUAL) \wedge \\
((ap\ c\_2EternaryComparisons\_2Enum2ordering\ (ap\ \mathsf{c\_2Earithmetic\_2ENUMERAL} \\
(ap\ \mathsf{c\_2Earithmetic\_2EBIT2}\ \mathsf{c\_2Earithmetic\_2EZERO}))) = c\_2EternaryComparisons\_2EGREATER)))))))
\end{array}
\tag{80}
$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((\forall V0l1 \in (ty\_2Elist\_2Elist\ A\_27a).(\forall V1a\_\_lt \in ((2^{A\_27a})^{A\_27a}).((ap\ (ap\ (ap\ (c\_2EternaryComparisons\_2Elist\_\_merge\ A\_27a)\ V1a\_\_lt)\ V0l1)\ (c\_2Elist\_2ENIL\ A\_27a)) = V0l1))) \wedge ((\forall V2v5 \in (ty\_2Elist\_2Elist\ A\_27a).(\forall V3v4 \in A\_27a.(\forall V4a\_\_lt \in ((2^{A\_27a})^{A\_27a}).((ap\ (ap\ (ap\ (c\_2EternaryComparisons\_2Elist\_\_merge\ A\_27a)\ V4a\_\_lt)\ (c\_2Elist\_2ENIL\ A\_27a))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V3v4)\ V2v5)) = (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V3v4)\ V2v5))))) \wedge (\forall V5y \in A\_27a.(\forall V6x \in A\_27a.(\forall V7l2 \in (ty\_2Elist\_2Elist\ A\_27a).(\forall V8l1 \in (ty\_2Elist\_2Elist\ A\_27a).(\forall V9a\_\_lt \in ((2^{A\_27a})^{A\_27a}).((ap\ (ap\ (ap\ (c\_2EternaryComparisons\_2Elist\_\_merge\ A\_27a)\ V9a\_\_lt)\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V6x)\ V8l1))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V5y)\ V7l2)) = (ap\ (ap\ (ap\ (\mathsf{c\_2Ebool\_2ECOND}\ (ty\_2Elist\_2Elist\ A\_27a))\ (ap\ (ap\ V9a\_\_lt\ V6x)\ V5y))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V6x)\ (ap\ (ap\ (ap\ (c\_2EternaryComparisons\_2Elist\_\_merge\ A\_27a)\ V9a\_\_lt)\ V8l1)\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V5y)\ V7l2))))\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V5y)\ (ap\ (ap\ (ap\ (c\_2EternaryComparisons\_2Elist\_\_merge\ A\_27a)\ V9a\_\_lt)\ (ap\ (ap\ (c\_2Elist\_2ECONS\ A\_27a)\ V6x)\ V8l1))\ V7l2)))))))))) \quad (81)$$

Theorem 1

$$(\forall V0a \in ty\_2Efrac\_2Efrac.(\forall V1b \in ty\_2Efrac\_2Efrac.((ap\ (ap\ \mathsf{c\_2Efrac\_2Efrac\_\_mul}\ V0a)\ V1b) = (ap\ (ap\ \mathsf{c\_2Efrac\_2Efrac\_\_mul}\ V1b)\ V0a))))$$