

thm\_2Efrac\_2EFRAC\_\_SGN\_\_MUL2  
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**Definition 1** We define  $c_2Emin_2E_3D$  to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c_2Ebool_2ET$  to be  $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Let  $ty_2Einteger_2Eint : \iota$  be given. Assume the following.

$$nonempty\ ty_2Einteger_2Eint \quad (1)$$

Let  $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod \\ A0\ A1) \end{aligned} \quad (2)$$

Let  $ty_2Efrac_2Efrac : \iota$  be given. Assume the following.

$$nonempty\ ty_2Efrac_2Efrac \quad (3)$$

Let  $c_2Efrac_2Erep_2Efrac : \iota$  be given. Assume the following.

$$c_2Efrac_2Erep_2Efrac \in ((ty_2Epair_2Eprod\ ty_2Einteger_2Eint \\ ty_2Einteger_2Eint)^{ty_2Efrac_2Efrac}) \quad (4)$$

Let  $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A_{27a}.nonempty\ A_{27a} \Rightarrow \forall A_{27b}.nonempty\ A_{27b} \Rightarrow c_2Epair_2EFST \\ A_{27a}\ A_{27b} \in (A_{27a}(ty_2Epair_2Eprod\ A_{27a}\ A_{27b})) \end{aligned} \quad (5)$$

**Definition 3** We define  $c_2Ebool_2E_21$  to be  $\lambda A_{27a} : \iota. (\lambda V0P \in (2^{A_{27a}}). (ap (ap (c_2Emin_2E_3D (2^{A_{27a}}))$

**Definition 4** We define  $c_2Efrac_2Efrac_2Efrac_nmr$  to be  $\lambda V0f \in ty_2Efrac_2Efrac. (ap (c_2Epair_2EFST\ ty_2Efrac_2Efrac\ f))$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (6)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$\text{nonempty } ty\_2Enum\_2Enum \quad (7)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (8)$$

**Definition 5** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

**Definition 6** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (9)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (10)$$

**Definition 7** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num\ m)$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (11)$$

**Definition 8** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic\_2E\_2B\ n)\ 0)$

**Definition 9** We define  $c\_2Earithmetic\_2ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (12)$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2Eint}) \quad (13)$$

**Definition 10** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p \ (ap\ P\ x)) \text{ then } (\lambda x.x \in A \wedge p) \text{ of type } \iota \Rightarrow \iota$ .

**Definition 11** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (c\_2Emin\_2E\_40\ (t\ a)))$

Let  $c\_2Einteger\_2Etint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_neg \in ((ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum)} \quad (14)$$

Let  $c\_2Einteger\_2Etint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum)})^{(2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})} \quad (15)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint)^{(2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})^{(2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})}} \quad (16)$$

**Definition 12** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)}$

**Definition 13** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap c\_2Einteger\_2Eint\_eq V0T1)$

Let  $c\_2Einteger\_2Etint\_lt : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_lt \in ((2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum)})^{(2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})} \quad (17)$$

**Definition 14** We define  $c\_2Einteger\_2Eint\_lt$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint.(ap c\_2Einteger\_2Eint\_lt V0T1 V1T2)$

**Definition 15** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 16** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 17** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.inj\_o (p V0t1 \Rightarrow p V1t2))))$

**Definition 18** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.(ap (c\_2Ebool\_2E\_21 2) (\lambda V3t3 \in A\_27a.inj\_o (p V1t1 \Rightarrow p V2t2))))))$

**Definition 19** We define  $c\_2EintExtension\_2ESGN$  to be  $\lambda V0x \in ty\_2Einteger\_2Eint.(ap (ap (ap (c\_2Ebool\_2E\_21 2) (\lambda V1y \in 2.V0y))) (c\_2EintExtension\_2ESGN V0x)))$

**Definition 20** We define  $c\_2Efrac\_2Efrac\_sgn$  to be  $\lambda V0f1 \in ty\_2Efrac\_2Efrac.(ap c\_2EintExtension\_2ESGN V0f1)$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2ESND A\_27a A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \quad (18)$$

**Definition 21** We define  $c\_2Efrac\_2Efrac\_dnm$  to be  $\lambda V0f \in ty\_2Efrac\_2Efrac.(ap (c\_2Epair\_2ESND t f))$

Let  $c\_2Einteger\_2Etint\_mul : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_mul \in (((ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)} \quad (19)$$



Assume the following.

$$\begin{aligned} & (\forall V0f \in ty\_2Efrac\_2Efrac.(p (ap (ap c\_2Einteger\_2Eint\_lt \\ & (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) (ap c\_2Efrac\_2Efrac\_dnm \\ & V0f)))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & (\forall V0a \in ty\_2Einteger\_2Eint.(\forall V1b \in ty\_2Einteger\_2Eint. \\ & ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) V1b)) \Rightarrow ((ap c\_2Efrac\_2Efrac\_nrm (ap c\_2Efrac\_2Eabs\_frac \\ & (ap (ap (c\_2Epair\_2E\_2C ty\_2Einteger\_2Eint ty\_2Einteger\_2Eint) \\ & V0a) V1b))) = V0a)))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} & (\forall V0a \in ty\_2Einteger\_2Eint.(\forall V1b \in ty\_2Einteger\_2Eint. \\ & ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) V0a)) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) V1b)) \Rightarrow (p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_mul V0a) V1b))))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint. \\ & ((ap c\_2EintExtension\_2ESGN (ap (ap c\_2Einteger\_2Eint\_mul V0x) \\ & V1y)) = (ap (ap c\_2Einteger\_2Eint\_mul (ap c\_2EintExtension\_2ESGN \\ & V0x)) (ap c\_2EintExtension\_2ESGN V1y)))))) \end{aligned} \quad (32)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (33)$$

Assume the following.

$$(\forall V0A \in 2.((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (34)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\ ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (35)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\ ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))))) \quad (36)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (37)$$

Assume the following.

$$\begin{aligned}
 & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\
 & (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg \\
 & p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\
 & ((\neg(p V1q)) \vee (\neg(p V0p)))))))))))
 \end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\
 & (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge (( \\
 & \neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))))))
 \end{aligned} \tag{39}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\
 (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p))))))) \tag{40}$$

### Theorem 1

$$\begin{aligned}
 & (\forall V0f1 \in ty\_2Efrac\_2Efrac. (\forall V1f2 \in ty\_2Efrac\_2Efrac. \\
 & ((ap c\_2Efrac\_2Efrac\_sgn (ap (ap c\_2Efrac\_2Efrac\_mul V0f1) \\
 & V1f2)) = (ap (ap c\_2Einteger\_2Eint\_mul (ap c\_2Efrac\_2Efrac\_sgn \\
 & V0f1)) (ap c\_2Efrac\_2Efrac\_sgn V1f2))))))
 \end{aligned}$$