

thm_2Ehtrat_2ETRAT__ADD__WELLDEFINED
(TMTkZB2fJTG8K7PdaEAu29DVUek7vBUB41v)

October 26, 2020

Definition 1 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.$ if $(\exists x \in A.p (ap P x))$ then (the $(\lambda x.x \in A \wedge p$
of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$
of type $\iota \Rightarrow \iota$.

Definition 3 We define $c_2Ebool_2E_3F$ to be $\lambda A.27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap V0P (ap (c_2Emin_2E_40 A$

Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in \omega \tag{1}$$

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \tag{2}$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{\omega}) \tag{3}$$

Definition 4 We define c_2Enum_2E0 to be $(ap\ c_2Enum_2EABS_num\ c_2Enum_2EZERO_REP)$.

Definition 5 We define $c_2Earithmic_2EZERO$ to be c_2Enum_2E0 .

Let $c_2Enum_2EREP_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (\omega^{ty_2Enum_2Enum}) \tag{4}$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (\omega^{\omega}) \tag{5}$$

Definition 6 We define c_2Ebool_2ET to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 7 We define $c_2Ebool_2E_21$ to be $\lambda A.27a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c_2Emin_2E_3D (2^{A-27a}$

Definition 8 We define c_2Enum_2ESUC to be $\lambda V0m \in ty_2Enum_2Enum.(ap\ c_2Enum_2EABS_num$

Let $c_2Earithmetic_2E_2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2B \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (6)$$

Definition 9 We define $c_2Earithmetic_2EBIT1$ to be $\lambda V0n \in ty_2Enum_2Enum.(ap\ (ap\ c_2Earithmetic_2E_2B$

Definition 10 We define $c_2Earithmetic_2ENUMERAL$ to be $\lambda V0x \in ty_2Enum_2Enum.V0x$.

Definition 11 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o\ (p\ P \Rightarrow p\ Q)$ of type ι .

Definition 12 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in 2.V0t))$

Definition 13 We define c_2Ebool_2EF to be $(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V0t \in 2.V0t))$.

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (7)$$

Let $c_2Ehrat_2Etrat_add : \iota$ be given. Assume the following.

$$c_2Ehrat_2Etrat_add \in (((ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum)})^{ty_2Enum_2Enum}) \quad (8)$$

Let $c_2Earithmetic_2E_2A : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2A \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \quad (9)$$

Let $c_2Ehrat_2Etrat_eq : \iota$ be given. Assume the following.

$$c_2Ehrat_2Etrat_eq \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum)}) \quad (10)$$

Definition 14 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap\ (ap\ c_2Emin_2E_3D_3D_3E\ V0t)\ c_2Ebool_2E_5C_2F$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2ESND\ A_27a\ A_27b \in (A_27b^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (11)$$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EFST\ A_27a\ A_27b \in (A_27a^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (12)$$

Definition 15 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in 2.V0t))$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod \\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \end{aligned} \quad (13)$$

Definition 16 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0x \in A_27a. \lambda V1y \in A_27b. (ap\ (c_2Epair_2E_2C\ V0x\ V1y))$

Definition 17 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A_27a. (\lambda V2t2 \in A_27a. (ap\ (c_2Ebool_2ECOND\ V0t\ V1t1\ V2t2))))$

Definition 18 We define $c_2Eprim_rec_2EPRE$ to be $\lambda V0m \in ty_2Enum_2Enum. (ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ V0m\ V0m)\ V0m)\ V0m)\ V0m)$

Assume the following.

$$\begin{aligned} (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\\ (ap\ (ap\ c_2Earithmetic_2E_2B\ c_2Enum_2E0\ V0m) = V0m) \wedge ((ap\ (\\ ap\ c_2Earithmetic_2E_2B\ V0m)\ c_2Enum_2E0) = V0m) \wedge ((ap\ (ap\ c_2Earithmetic_2E_2B \\ (ap\ c_2Enum_2ESUC\ V0m))\ V1n) = (ap\ c_2Enum_2ESUC\ (ap\ (ap\ c_2Earithmetic_2E_2B \\ V0m)\ V1n))) \wedge ((ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ (ap\ c_2Enum_2ESUC \\ V1n)) = (ap\ c_2Enum_2ESUC\ (ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ V1n)))))) \end{aligned} \quad (14)$$

Assume the following.

$$(\forall V0m \in ty_2Enum_2Enum. ((V0m = c_2Enum_2E0) \vee (\exists V1n \in ty_2Enum_2Enum. (V0m = (ap\ c_2Enum_2ESUC\ V1n)))))) \quad (15)$$

Assume the following.

$$\begin{aligned} (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\\ ((ap\ (ap\ c_2Earithmetic_2E_2A\ c_2Enum_2E0\ V0m) = c_2Enum_2E0) \wedge \\ (((ap\ (ap\ c_2Earithmetic_2E_2A\ V0m)\ c_2Enum_2E0) = c_2Enum_2E0) \wedge \\ (((ap\ (ap\ c_2Earithmetic_2E_2A\ (ap\ c_2Earithmetic_2ENUMERAL \\ (ap\ c_2Earithmetic_2EBIT1\ c_2Earithmetic_2EZERO)))\ V0m) = V0m) \wedge \\ (((ap\ (ap\ c_2Earithmetic_2E_2A\ V0m)\ (ap\ c_2Earithmetic_2ENUMERAL \\ (ap\ c_2Earithmetic_2EBIT1\ c_2Earithmetic_2EZERO))) = V0m) \wedge \\ ((ap\ (ap\ c_2Earithmetic_2E_2A\ (ap\ c_2Enum_2ESUC\ V0m))\ V1n) = (ap \\ (ap\ c_2Earithmetic_2E_2B\ (ap\ (ap\ c_2Earithmetic_2E_2A\ V0m)\ V1n)) \\ V1n)) \wedge ((ap\ (ap\ c_2Earithmetic_2E_2A\ V0m)\ (ap\ c_2Enum_2ESUC\ V1n)) = \\ (ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ (ap\ (ap\ c_2Earithmetic_2E_2A \\ V0m)\ V1n)))))) \end{aligned} \quad (16)$$

Assume the following.

$$(ap\ (ap\ c_2Earithmetic_2E_2A\ V0m)\ V1n) = (ap\ (ap\ c_2Earithmetic_2E_2A\ V1n)\ V0m)) \quad (17)$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\\
& \quad \forall V2p \in ty_2Enum_2Enum. ((ap (ap c_2Earithmetic_2E_2A (ap \\
& (ap c_2Earithmetic_2E_2B V0m) V1n)) V2p) = (ap (ap c_2Earithmetic_2E_2B \\
& (ap (ap c_2Earithmetic_2E_2A V0m) V2p)) (ap (ap c_2Earithmetic_2E_2A \\
& \quad V1n) V2p))))))
\end{aligned} \tag{18}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\\
& \quad \forall V2p \in ty_2Enum_2Enum. ((ap (ap c_2Earithmetic_2E_2A V0m) \\
& (ap (ap c_2Earithmetic_2E_2A V1n) V2p)) = (ap (ap c_2Earithmetic_2E_2A \\
& (ap (ap c_2Earithmetic_2E_2A V0m) V1n)) V2p))))))
\end{aligned} \tag{19}$$

Assume the following.

$$True \tag{20}$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p \\
V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \tag{21}$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \tag{22}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\
& (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\
& (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t))))))
\end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2. (((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge \\
& (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee \\
& (p V0t)) \Leftrightarrow (p V0t))))))
\end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2. (((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\
& True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((\\
& (p V0t) \Rightarrow False) \Leftrightarrow \neg (p V0t))))))
\end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0t \in 2. ((\neg (\neg (p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\
& ((\neg False) \Leftrightarrow True))))
\end{aligned} \tag{26}$$

Assume the following.

$$\forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in A.27a.((V0x = V0x) \Leftrightarrow True)) \quad (27)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg(\\ & p\ V0t)))))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty_2Enum_2Enum.(\forall V1y \in ty_2Enum_2Enum.(\\ & \forall V2x_27 \in ty_2Enum_2Enum.(\forall V3y_27 \in ty_2Enum_2Enum. \\ & ((ap\ (ap\ c.2Ehrat_2Etrat_add\ (ap\ (ap\ (c.2Epair_2E_2C\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)\ V0x)\ V1y))\ (ap\ (ap\ (c.2Epair_2E_2C\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)\ V2x_27)\ V3y_27)) = (ap\ (ap\ (c.2Epair_2E_2C\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)\ (ap\ c.2Eprim_rec_2EPRE\ (ap\ (ap\ c.2Earithmetic_2E_2B \\ & (ap\ (ap\ c.2Earithmetic_2E_2A\ (ap\ c.2Enum_2ESUC\ V0x))\ (ap\ c.2Enum_2ESUC \\ & V3y_27)))\ (ap\ (ap\ c.2Earithmetic_2E_2A\ (ap\ c.2Enum_2ESUC\ V2x_27)) \\ & (ap\ c.2Enum_2ESUC\ V1y))))))\ (ap\ c.2Eprim_rec_2EPRE\ (ap\ (ap\ c.2Earithmetic_2E_2A \\ & (ap\ c.2Enum_2ESUC\ V1y))\ (ap\ c.2Enum_2ESUC\ V3y_27)))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in ty_2Enum_2Enum.(\forall V1y \in ty_2Enum_2Enum.(\\ & \forall V2x_27 \in ty_2Enum_2Enum.(\forall V3y_27 \in ty_2Enum_2Enum. \\ & ((p\ (ap\ (ap\ c.2Ehrat_2Etrat_eq\ (ap\ (ap\ (c.2Epair_2E_2C\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)\ V0x)\ V1y))\ (ap\ (ap\ (c.2Epair_2E_2C\ ty_2Enum_2Enum \\ & ty_2Enum_2Enum)\ V2x_27)\ V3y_27))) \Leftrightarrow ((ap\ (ap\ c.2Earithmetic_2E_2A \\ & (ap\ c.2Enum_2ESUC\ V0x))\ (ap\ c.2Enum_2ESUC\ V3y_27)) = (ap\ (ap\ c.2Earithmetic_2E_2A \\ & (ap\ c.2Enum_2ESUC\ V2x_27))\ (ap\ c.2Enum_2ESUC\ V1y)))))) \end{aligned} \quad (30)$$

Assume the following.

$$(\forall V0n \in ty_2Enum_2Enum.(\neg((ap\ c.2Enum_2ESUC\ V0n) = c.2Enum_2E0))) \quad (31)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty\ A.27a \Rightarrow \forall A.27b.nonempty\ A.27b \Rightarrow (\\ & \forall V0x \in (ty_2Epair_2Eprod\ A.27a\ A.27b).((ap\ (ap\ (c.2Epair_2E_2C \\ & A.27a\ A.27b)\ (ap\ (c.2Epair_2EFST\ A.27a\ A.27b)\ V0x))\ (ap\ (c.2Epair_2ESND \\ & A.27a\ A.27b)\ V0x)) = V0x)) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & (((ap\ c.2Eprim_rec_2EPRE\ c.2Enum_2E0) = c.2Enum_2E0) \wedge (\forall V0m \in \\ & ty_2Enum_2Enum.((ap\ c.2Eprim_rec_2EPRE\ (ap\ c.2Enum_2ESUC\ V0m)) = \\ & V0m))) \end{aligned} \quad (33)$$

Theorem 1

$(\forall V0p \in (ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)).$
 $(\forall V1q \in (ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)).$
 $(\forall V2r \in (ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)).$
 $((p\ (ap\ (ap\ c_2Ehrat_2Etrat_eq\ V0p)\ V1q)) \Rightarrow (p\ (ap\ (ap\ c_2Ehrat_2Etrat_eq\ (ap\ (ap\ c_2Ehrat_2Etrat_add\ V0p)\ V2r))\ (ap\ (ap\ c_2Ehrat_2Etrat_add\ V1q)\ V2r))))))$