thm_2Ehreal_2ECUT__NEARTOP__MUL (TM-RvmBKDfM3oBGaoqpfsyGafYWQqApsHQQA)

October 26, 2020

Definition 1 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 2 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 3 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 4 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27a}$

Definition 5 *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Definition 6 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{1}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{2}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{3}$$

Definition 7 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 8 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{4}$$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \tag{5}$$

Definition 9 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2E$

Definition 10 *We define* c_2Ehrat_2Etrat__1 *to be* $(ap\ (ap\ ($c_2Epair_2E_2C $ty\_2Enum\_2Enum\ ty\_2Enum\_$

Let $c\_2Ehrat\_2Etrat\_\_eq : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2} \quad (6)$$

Let $ty\_2Ehrat\_2Ehrat : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehrat\_2Ehrat \quad (7)$$

Let $c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS \in (ty\_2Ehrat\_2Ehrat^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})}) \quad (8)$$

Definition 11 *We define* c_2Ehrat_2Ehrat__ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2$

Definition 12 *We define* c_2Ehrat_2Ehrat__1 *to be* $(ap$ c_2Ehrat_2Ehrat__ABS c_2Ehrat_2Etrat__1$)$.

Let $c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Ehrat\_2Ehra} \quad (9)$$

Definition 13 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 14 *We define* c_2Ehrat_2Ehrat__REP *to be* $\lambda V0a \in ty_2Ehrat_2Ehrat.(ap\ ($c_2Emin_2E_40 $(ty\_2$

Let $c\_2Ehrat\_2Etrat\_\_mul : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty} \quad (10)$$

Definition 15 *We define* c_2Ehrat_2Ehrat__mul *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2E$

Let $c\_2Ehrat\_2Etrat\_\_add : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty} \quad (11)$$

Definition 16 *We define* c_2Ehrat_2Ehrat__add *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2E$

Definition 17 *We define* c_2Ebool_2E_3F *to be* $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ V0P\ (ap\ ($c_2Emin_2E_40

Definition 18 *We define* c_2Ehreal_2Ehrat__lt *to be* $\lambda V0x \in ty\_2Ehrat\_2Ehrat.\lambda V1y \in ty\_2Ehrat\_2Ehrat$

Definition 19 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap$ c_2Emin_2E_3D_3D_3E $V0t)$ c_2Ebool_2E

Let $ty\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{12}$$

Let $c\_2Ehreal\_2Ecut : \iota$ be given. Assume the following.

$$c\_2Ehreal\_2Ecut \in ((2^{ty\_2Ehrat\_2Ehrat})^{ty\_2Ehreal\_2Ehreal}) \tag{13}$$

Assume the following.

$$True \tag{14}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{15}$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \tag{16}$$

Assume the following.

$$(\forall V0t \in 2.((p\ V0t) \vee (\neg (p\ V0t)))) \tag{17}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge ((((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t))))))) \tag{18}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow False) \Leftrightarrow (\neg (p\ V0t)))))))) \tag{19}$$

Assume the following.

$$((\forall V0t \in 2.((\neg (\neg (p\ V0t))) \Leftrightarrow (p\ V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \tag{20}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg (p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (p\ V0t))))))) \tag{21}$$

Assume the following.

$$(\forall V0h \in ty\_2Ehrat\_2Ehrat.((ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ \mathsf{c\_2Ehrat\_2Ehrat\_\_1})\ V0h) = V0h)) \tag{22}$$

Assume the following.

$$(\forall V0x \in ty\_2Ehrat\_2Ehrat.(\forall V1y \in ty\_2Ehrat\_2Ehrat.(\forall V2z \in ty\_2Ehrat\_2Ehrat.((ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_add}\ V0x)\ V1y))\ V2z) = (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_add}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V0x)\ V2z))\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V1y)\ V2z)))))) \quad (23)$$

Assume the following.

$$(\forall V0x \in ty\_2Ehrat\_2Ehrat.(\forall V1y \in ty\_2Ehrat\_2Ehrat.(\forall V2z \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_add}\ V2z)\ V0x))\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_add}\ V2z)\ V1y)))\Leftrightarrow(p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V0x)\ V1y)))))) \quad (24)$$

Assume the following.

$$(\forall V0x \in ty\_2Ehrat\_2Ehrat.(\forall V1y \in ty\_2Ehrat\_2Ehrat.(\forall V2z \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_add}\ V0x)\ V2z))\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_add}\ V1y)\ V2z)))\Leftrightarrow(p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V0x)\ V1y)))))) \quad (25)$$

Assume the following.

$$(\forall V0x \in ty\_2Ehrat\_2Ehrat.(\forall V1y \in ty\_2Ehrat\_2Ehrat.(\forall V2z \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V2z)\ V0x))\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V2z)\ V1y)))\Leftrightarrow(p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V0x)\ V1y)))))) \quad (26)$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\exists V1x \in ty\_2Ehrat\_2Ehrat.(p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V1x)))) \quad (27)$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1x \in ty\_2Ehrat\_2Ehrat.(\forall V2y \in ty\_2Ehrat\_2Ehrat.(((\neg(p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V1x)))\wedge(p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V1x)\ V2y)))\Rightarrow(\neg(p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V2y))))))) \quad (28)$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1x \in ty\_2Ehrat\_2Ehrat.(\forall V2y \in ty\_2Ehrat\_2Ehrat.(((p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V1x))\wedge(\neg(p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V2y))))\Rightarrow(p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V1x)\ V2y)))))) \quad (29)$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1e \in ty\_2Ehrat\_2Ehrat.(\exists V2x \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V2x))\wedge(\neg(p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_add}\ V2x)\ V1e))))))))) \tag{30}$$

Theorem 1

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1u \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ \mathsf{c\_2Ehrat\_2Ehrat\_\_1})\ V1u))\Rightarrow(\exists V2x \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V2x))\wedge(\neg(p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V1u)\ V2x))))))))))$$