thm_2Ehreal_2EHREAL__MUL__LID (TMdbZRP5wrzQrz2mAzmKZ3HymNvWdv5BAYc)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 3 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 4 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 5 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 6 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2EF}$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{2}$$

Let $ty\_2Ehrat\_2Ehrat : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehrat\_2Ehrat \tag{3}$$

Let $c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Ehrat\_2Ehra} \tag{4}$$

Definition 7 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 8 *We define* c_2Ehrat_2Ehrat__REP *to be* $\lambda V0a \in ty\_2Ehrat\_2Ehrat.(ap\ (\mathsf{c\_2Emin\_2E\_40}\ (ty\_2E$

Let $c\_2Ehrat\_2Etrat\_\_inv : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_inv \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2}\tag{5}$$

Let $c\_2Ehrat\_2Etrat\_\_eq : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2E}\tag{6}$$

Let $c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Ehrat\_\_ABS\_\_CLASS \in (ty\_2Ehrat\_2Ehrat^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})})\tag{7}$$

Definition 9 *We define* c_2Ehrat_2Ehrat__ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2E$

Definition 10 *We define* c_2Ehrat_2Ehrat__inv *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.(ap$ c_2Ehrat_2Ehrat__ABS

Let $c\_2Ehrat\_2Etrat\_\_add : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty}\tag{8}$$

Definition 11 *We define* c_2Ehrat_2Ehrat__add *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2E$

Definition 12 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap$ (c_2Emin_2E_40

Definition 13 *We define* c_2Ehreal_2Ehrat__lt *to be* $\lambda V0x \in ty\_2Ehrat\_2Ehrat.\lambda V1y \in ty\_2Ehrat\_2Ehrat$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega\tag{9}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega})\tag{10}$$

Definition 14 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 15 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap$ (c_2Ebool_2E_21 2) $(\lambda V2t \in$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})})\tag{11}$$

Definition 16 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Definition 17 *We define* c_2Ehrat_2Etrat__1 *to be* $(ap\ (ap$ (c_2Epair_2E_2C $ty\_2Enum\_2Enum\ ty\_2Enum\_$

Definition 18 *We define* c_2Ehrat_2Ehrat__1 *to be* $(ap$ c_2Ehrat_2Ehrat__ABS c_2Ehrat_2Etrat__1).

Definition 19 *We define* c_2Ehreal_2Ecut__of__hrat *to be* $\lambda V0x \in ty\_2Ehrat\_2Ehrat.(\lambda V1y \in ty\_2Ehrat\_2$

Let $ty\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Ehreal\_2Ehreal \tag{12}$$

Let $c\_2Ehreal\_2Ehreal : \iota$ be given. Assume the following.

$$c\_2Ehreal\_2Ehreal \in (ty\_2Ehreal\_2Ehreal^{(2^{ty\_2Ehrat\_2Ehrat})}) \tag{13}$$

Definition 20 *We define* c_2Ehreal_2Ehreal__1 *to be* $(ap\ c\_2Ehreal\_2Ehreal\ (ap$ c_2Ehreal_2Ecut__of__hrat

Let $c\_2Ehreal\_2Ecut : \iota$ be given. Assume the following.

$$c\_2Ehreal\_2Ecut \in ((2^{ty\_2Ehrat\_2Ehrat})^{ty\_2Ehreal\_2Ehreal}) \tag{14}$$

Let $c\_2Ehrat\_2Etrat\_\_mul : \iota$ be given. Assume the following.

$$c\_2Ehrat\_2Etrat\_\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty} \tag{15}$$

Definition 21 *We define* c_2Ehrat_2Ehrat__mul *to be* $\lambda V0T1 \in ty\_2Ehrat\_2Ehrat.\lambda V1T2 \in ty\_2Ehrat\_2E$

Definition 22 *We define* c_2Ehreal_2Ehreal__mul *to be* $\lambda V0X \in ty\_2Ehreal\_2Ehreal.\lambda V1Y \in ty\_2Ehreal\_2$

Definition 23 *We define* c_2Ehreal_2Eisacut *to be* $\lambda V0C \in (2^{ty\_2Ehrat\_2Ehrat}).(ap\ (ap$ c_2Ebool_2E_2F_5C (

Assume the following.

$$True \tag{16}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{17}$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge ((((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t))))))) \tag{18}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \tag{19}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \tag{20}$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg (p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg (p\ V0t))))))) \tag{21}$$

Assume the following.

$$(\forall V0h \in ty\_2Ehrat\_2Ehrat.(\forall V1i \in ty\_2Ehrat\_2Ehrat. ((ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V0h)\ V1i) = (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V1i)\ V0h)))) \tag{22}$$

Assume the following.

$$(\forall V0h \in ty\_2Ehrat\_2Ehrat.(\forall V1i \in ty\_2Ehrat\_2Ehrat. (\forall V2j \in ty\_2Ehrat\_2Ehrat.((ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V0h)\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V1i)\ V2j)) = (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V0h)\ V1i))\ V2j))))) \tag{23}$$

Assume the following.

$$(\forall V0h \in ty\_2Ehrat\_2Ehrat.((ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_inv}\ V0h))\ V0h) = \mathsf{c\_2Ehrat\_2Ehrat\_\_1})) \tag{24}$$

Assume the following.

$$(\forall V0x \in ty\_2Ehrat\_2Ehrat.((ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V0x)\ \mathsf{c\_2Ehrat\_2Ehrat\_\_1}) = V0x)) \tag{25}$$

Assume the following.

$$(\forall V0x \in ty\_2Ehrat\_2Ehrat.(\forall V1y \in ty\_2Ehrat\_2Ehrat. ((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V0x)\ V1y))\ V1y)) \Leftrightarrow (p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V0x)\ \mathsf{c\_2Ehrat\_2Ehrat\_\_1}))))) \tag{26}$$

Assume the following.

$$(\forall V0x \in ty\_2Ehrat\_2Ehrat.(\forall V1y \in ty\_2Ehrat\_2Ehrat. ((p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_inv}\ V0x))\ V1y))\ \mathsf{c\_2Ehrat\_2Ehrat\_\_1})) \Leftrightarrow (p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V1y)\ V0x))))) \tag{27}$$

Assume the following.

$$(\forall V0h \in ty\_2Ehrat\_2Ehrat.(p\ (ap\ \mathsf{c\_2Ehreal\_2Eisacut}\ (ap\ \mathsf{c\_2Ehreal\_2Ecut\_\_of\_\_hrat}\ V0h)))) \tag{28}$$

Assume the following.

$$((\forall V0a \in ty\_2Ehreal\_2Ehreal.((ap\ c\_2Ehreal\_2Ehreal\ (ap\ c\_2Ehreal\_2Ecut\ V0a)) = V0a)) \wedge (\forall V1r \in (2^{ty\_2Ehrat\_2Ehrat}). ((p\ (ap\ \mathsf{c\_2Ehreal\_2Eisacut}\ V1r)) \Leftrightarrow ((ap\ c\_2Ehreal\_2Ecut\ (ap\ c\_2Ehreal\_2Ehreal\ V1r)) = V1r)))) \tag{29}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.(((ap\ c\_2Ehreal\_2Ecut\ V0X) = (ap\ c\_2Ehreal\_2Ecut\ V1Y)) \Rightarrow (V0X = V1Y)))) \tag{30}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1x \in ty\_2Ehrat\_2Ehrat.(\forall V2y \in ty\_2Ehrat\_2Ehrat.(((p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V1x)) \wedge (p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V2y)\ V1x))) \Rightarrow (p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V2y)))))) \tag{31}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1x \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V1x)) \Rightarrow (\exists V2y \in ty\_2Ehrat\_2Ehrat.((p\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V2y)) \wedge (p\ (ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehrat\_\_lt}\ V1x)\ V2y)))))))) \tag{32}$$

Assume the following.

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.(\forall V1Y \in ty\_2Ehreal\_2Ehreal.(p\ (ap\ \mathsf{c\_2Ehreal\_2Eisacut}\ (\lambda V2w \in ty\_2Ehrat\_2Ehrat.(ap\ (\mathsf{c\_2Ebool\_2E\_3F}\ ty\_2Ehrat\_2Ehrat)\ (\lambda V3x \in ty\_2Ehrat\_2Ehrat.(ap\ (\mathsf{c\_2Ebool\_2E\_3F}\ ty\_2Ehrat\_2Ehrat)\ (\lambda V4y \in ty\_2Ehrat\_2Ehrat.(ap\ (ap\ \mathsf{c\_2Ebool\_2E\_2F\_5C}\ (ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ ty\_2Ehrat\_2Ehrat)\ V2w)\ (ap\ (ap\ \mathsf{c\_2Ehrat\_2Ehrat\_\_mul}\ V3x)\ V4y)))\ (ap\ (ap\ \mathsf{c\_2Ebool\_2E\_2F\_5C}\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V0X)\ V3x))\ (ap\ (ap\ c\_2Ehreal\_2Ecut\ V1Y)\ V4y)))))))))))) \tag{33}$$

Theorem 1

$$(\forall V0X \in ty\_2Ehreal\_2Ehreal.((ap\ (ap\ \mathsf{c\_2Ehreal\_2Ehreal\_\_mul}\ \mathsf{c\_2Ehreal\_2Ehreal\_\_1})\ V0X) = V0X))$$