

# thm\_2Eint\_\_arith\_2Ecooper\_\_lemma\_\_1 (TMJ2ngZBUpp3PigDQdSv8bTfKG368QoNsQb)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let  $c\_2Earithmetic\_2E\_2A : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2A \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{2}$$

**Definition 3** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p))$  of type  $\iota \Rightarrow \iota$ .

**Definition 4** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A\_27a))))$

**Definition 5** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))))$

**Definition 6** We define  $c\_2Edivides\_2Edivides$  to be  $\lambda V0a \in ty\_2Enum\_2Enum.\lambda V1b \in ty\_2Enum\_2Enum$

**Definition 7** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 8** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2)))$

**Definition 9** We define  $c\_2Egcd\_2Eis\_gcd$  to be  $\lambda V0a \in ty\_2Enum\_2Enum.\lambda V1b \in ty\_2Enum\_2Enum.\lambda V2c \in ty\_2Enum\_2Enum$

Let  $c\_2Egcd\_2Egcd : \iota$  be given. Assume the following.

$$c\_2Egcd\_2Egcd \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \tag{3}$$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \tag{4}$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \tag{5}$$

**Definition 10** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $ty\_2Einteger\_2Eint : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \quad (6)$$

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (7)$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (8)$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2Eint}) \quad (9)$$

**Definition 11** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (c\_2Emin\_2E40\ (t$

Let  $c\_2Einteger\_2Eint\_mul : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (10)$$

Let  $c\_2Einteger\_2Eint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (11)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})}) \quad (12)$$

**Definition 12** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)$

**Definition 13** We define  $c\_2Einteger\_2Eint\_mul$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint$

**Definition 14** We define  $c\_2Einteger\_2Eint\_divides$  to be  $\lambda V0p \in ty\_2Einteger\_2Eint.\lambda V1q \in ty\_2Einteger\_2Eint$

Let  $c\_2Einteger\_2Eint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (13)$$

**Definition 15** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap\ c\_2Einteger\_2Eint$

Let  $c\_2Einteger\_2Etint\_add : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)} \tag{14}$$

**Definition 16** We define  $c\_2Einteger\_2Eint\_add$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint.$

**Definition 17** We define  $c\_2Einteger\_2Eint\_sub$  to be  $\lambda V0x \in ty\_2Einteger\_2Eint.\lambda V1y \in ty\_2Einteger\_2Eint.$

**Definition 18** We define  $c\_2Ebool\_2EF$  to be  $(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V0t \in 2.V0t))$ .

**Definition 19** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2.V2t))))$

**Definition 20** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap\ (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E\_7E))$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.((ap\ (ap\ c\_2Earithmetic\_2E\_2A\ V0m)\ V1n) = c\_2Enum\_2E0) \Leftrightarrow ((V0m = c\_2Enum\_2E0) \vee (V1n = c\_2Enum\_2E0)))) \tag{15}$$

Assume the following.

$$True \tag{16}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \tag{17}$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \tag{18}$$

Assume the following.

$$(\forall V0t \in 2.((p\ V0t) \vee (\neg(p\ V0t)))) \tag{19}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p\ V0t) \Leftrightarrow (p\ V0t)))) \tag{20}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p\ V0t1) \wedge ((p\ V1t2) \wedge (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \wedge (p\ V2t3)))))) \tag{21}$$

Assume the following.

$$(\forall V0t \in 2.(((p\ V0t) \Rightarrow False) \Rightarrow (\neg(p\ V0t)))) \tag{22}$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (23)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge \\ & (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee \\ & (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (26)$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \end{aligned} \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (28)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (29)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\ & p V0t)))))) \end{aligned} \quad (30)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\forall V1x \in A\_27a.(p (ap V0P V1x)))) \Leftrightarrow (\exists V2x \in A\_27a.(\neg(p (ap V0P V2x)))))) \quad (31)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\exists V1x \in A\_27a.(p (ap V0P V1x)))) \Leftrightarrow (\forall V2x \in A\_27a.(\neg(p (ap V0P V2x)))))) \quad (32)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in \\ & (2^{A.27a}).((\forall V2x \in A.27a.((p (ap V0P V2x)) \wedge (p (ap V1Q V2x)))) \Leftrightarrow \\ & ((\forall V3x \in A.27a.(p (ap V0P V3x))) \wedge (\forall V4x \in A.27a.(p ( \\ & \quad ap V1Q V4x))))))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in \\ & 2.((\forall V2x \in A.27a.(p (ap V0P V2x))) \wedge (p V1Q)) \Leftrightarrow (\forall V3x \in \\ & \quad A.27a.((p (ap V0P V3x)) \wedge (p V1Q)))))) \end{aligned} \quad (34)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).((p V0P) \wedge (\forall V2x \in A.27a.(p (ap V1Q V2x)))) \Leftrightarrow (\forall V3x \in \\ & \quad A.27a.((p V0P) \wedge (p (ap V1Q V3x)))))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).(((p V0P) \vee (\exists V2x \in A.27a.(p (ap V1Q V2x)))) \Leftrightarrow (\exists V3x \in \\ & \quad A.27a.((p V0P) \vee (p (ap V1Q V3x)))))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}).(\forall V1Q \in \\ & 2.((\exists V2x \in A.27a.((p (ap V0P V2x)) \wedge (p V1Q))) \Leftrightarrow ((\exists V3x \in \\ & \quad A.27a.(p (ap V0P V3x))) \wedge (p V1Q)))))) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).((\exists V2x \in A.27a.((p V0P) \wedge (p (ap V1Q V2x)))) \Leftrightarrow ((p \\ & \quad V0P) \wedge (\exists V3x \in A.27a.(p (ap V1Q V3x)))))) \end{aligned} \quad (38)$$

Assume the following.

$$\begin{aligned} & \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in ( \\ & 2^{A.27a}).((\forall V2x \in A.27a.((p V0P) \vee (p (ap V1Q V2x)))) \Leftrightarrow ((p \\ & \quad V0P) \vee (\forall V3x \in A.27a.(p (ap V1Q V3x)))))) \end{aligned} \quad (39)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V0A) \vee ( \\ & (p V1B) \vee (p V2C))) \Leftrightarrow (((p V0A) \vee (p V1B)) \vee (p V2C)))))) \end{aligned} \quad (40)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((p V0A) \vee (p V1B)) \Leftrightarrow ((p V1B) \vee \\ & \quad (p V0A)))) \end{aligned} \quad (41)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg(p V0A)) \vee (\neg(p V1B)))))) \wedge (((\neg((p V0A) \vee (p V1B))) \Leftrightarrow ((\neg(p V0A)) \wedge (\neg(p V1B))))))))) \quad (42)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (43)$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x_{.27} \in 2.(\forall V2y \in 2.(\forall V3y_{.27} \in 2.(((p V0x) \Leftrightarrow (p V1x_{.27})) \wedge ((p V1x_{.27}) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_{.27})))))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_{.27}) \Rightarrow (p V3y_{.27})))))) \quad (44)$$

Assume the following.

$$\forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow \forall A_{.27b}.nonempty\ A_{.27b} \Rightarrow (\forall V0P \in ((2^{A_{.27b}})^{A_{.27a}}).((\forall V1x \in A_{.27a}.(\exists V2y \in A_{.27b}.(p\ (ap\ (ap\ V0P\ V1x)\ V2y)))) \Leftrightarrow (\exists V3f \in (A_{.27b}^{A_{.27a}}).(\forall V4x \in A_{.27a}.(p\ (ap\ (ap\ V0P\ V4x)\ (ap\ V3f\ V4x))))))) \quad (45)$$

Assume the following.

$$\forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow (\forall V0f \in ((A_{.27a}^{A_{.27a}})^{A_{.27a}}).((\forall V1x \in A_{.27a}.(\forall V2y \in A_{.27a}.(\forall V3z \in A_{.27a}.((ap\ (ap\ V0f\ V1x)\ (ap\ (ap\ V0f\ V2y)\ V3z)) = (ap\ (ap\ V0f\ (ap\ (ap\ V0f\ V1x)\ V2y))\ V3z)))))) \Rightarrow ((\forall V4x \in A_{.27a}.(\forall V5y \in A_{.27a}.((ap\ (ap\ V0f\ V4x)\ V5y) = (ap\ (ap\ V0f\ V5y)\ V4x)))) \Rightarrow (\forall V6x \in A_{.27a}.(\forall V7y \in A_{.27a}.(\forall V8z \in A_{.27a}.((ap\ (ap\ V0f\ V6x)\ (ap\ (ap\ V0f\ V7y)\ V8z)) = (ap\ (ap\ V0f\ V7y)\ (ap\ (ap\ V0f\ V6x)\ V8z)))))))))) \quad (46)$$

Assume the following.

$$(\forall V0a \in ty\_2Enum\_2Enum.(\forall V1b \in ty\_2Enum\_2Enum.(p\ (ap\ (ap\ (ap\ c\_2Egcd\_2Eis\_gcd\ V0a)\ V1b)\ (ap\ (ap\ c\_2Egcd\_2Egcd\ V0a)\ V1b)))))) \quad (47)$$

Assume the following.

$$(\forall V0n \in ty\_2Enum\_2Enum.(\forall V1m \in ty\_2Enum\_2Enum.((ap\ (ap\ c\_2Egcd\_2Egcd\ V0n)\ V1m) = c\_2Enum\_2E0) \Leftrightarrow ((V0n = c\_2Enum\_2E0) \wedge (V1m = c\_2Enum\_2E0)))))) \quad (48)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.(\forall V2z \in ty\_2Einteger\_2Eint.((ap\ (ap\ c\_2Einteger\_2Eint\_sub\ V0x)\ (ap\ (ap\ c\_2Einteger\_2Eint\_sub\ V1y)\ V2z)) = (ap\ (ap\ c\_2Einteger\_2Eint\_sub\ (ap\ (ap\ c\_2Einteger\_2Eint\_add\ V0x)\ V2z))\ V1y)))))) \quad (49)$$

Assume the following.

$$\begin{aligned}
& (\forall V0n \in ty\_2Enum\_2Enum. (\forall V1m \in ty\_2Enum\_2Enum. ( \\
& (p (ap (ap c\_2Einteger\_2Eint\_divides (ap c\_2Einteger\_2Eint\_of\_num \\
& V0n)) (ap c\_2Einteger\_2Eint\_of\_num V1m))) \Leftrightarrow (p (ap (ap c\_2Edivides\_2Edivides \\
& V0n) V1m))))))
\end{aligned} \tag{50}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1q \in ty\_2Einteger\_2Eint. \\
& (\forall V2r \in ty\_2Einteger\_2Eint. ((\neg(V1q = (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0))) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_divides (ap (ap \\
& c\_2Einteger\_2Eint\_mul V0p) V1q)) (ap (ap c\_2Einteger\_2Eint\_mul \\
& V2r) V1q))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_divides V0p) V2r))))))
\end{aligned} \tag{51}$$

Assume the following.

$$\begin{aligned}
& (\forall V0y \in ty\_2Einteger\_2Eint. (\forall V1x \in ty\_2Einteger\_2Eint. \\
& ((ap (ap c\_2Einteger\_2Eint\_add V1x) V0y) = (ap (ap c\_2Einteger\_2Eint\_add \\
& V0y) V1x))))
\end{aligned} \tag{52}$$

Assume the following.

$$\begin{aligned}
& (\forall V0y \in ty\_2Einteger\_2Eint. (\forall V1x \in ty\_2Einteger\_2Eint. \\
& ((ap (ap c\_2Einteger\_2Eint\_mul V1x) V0y) = (ap (ap c\_2Einteger\_2Eint\_mul \\
& V0y) V1x))))
\end{aligned} \tag{53}$$

Assume the following.

$$\begin{aligned}
& (\forall V0z \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_add \\
& V2x) (ap (ap c\_2Einteger\_2Eint\_add V1y) V0z)) = (ap (ap c\_2Einteger\_2Eint\_add \\
& (ap (ap c\_2Einteger\_2Eint\_add V2x) V1y)) V0z))))))
\end{aligned} \tag{54}$$

Assume the following.

$$\begin{aligned}
& (\forall V0z \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_mul \\
& V2x) (ap (ap c\_2Einteger\_2Eint\_mul V1y) V0z)) = (ap (ap c\_2Einteger\_2Eint\_mul \\
& (ap (ap c\_2Einteger\_2Eint\_mul V2x) V1y)) V0z))))))
\end{aligned} \tag{55}$$

Assume the following.

$$\begin{aligned}
& (\forall V0z \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2x \in ty\_2Einteger\_2Eint. ((ap (ap c\_2Einteger\_2Eint\_mul \\
& V2x) (ap (ap c\_2Einteger\_2Eint\_add V1y) V0z)) = (ap (ap c\_2Einteger\_2Eint\_add \\
& (ap (ap c\_2Einteger\_2Eint\_mul V2x) V1y)) (ap (ap c\_2Einteger\_2Eint\_mul \\
& V2x) V0z))))))
\end{aligned} \tag{56}$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((ap (ap c\_2Einteger\_2Eint\_add (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) V0x) = V0x)) \quad (57)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.(\forall V2z \in ty\_2Einteger\_2Eint.((ap (ap c\_2Einteger\_2Eint\_mul (ap (ap c\_2Einteger\_2Eint\_add V0x) V1y)) V2z) = (ap (ap c\_2Einteger\_2Eint\_add (ap (ap c\_2Einteger\_2Eint\_mul V0x) V2z)) (ap (ap c\_2Einteger\_2Eint\_mul V1y) V2z))))))) \quad (58)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((ap (ap c\_2Einteger\_2Eint\_mul V0x) (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) = (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0))) \quad (59)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.((ap (ap c\_2Einteger\_2Eint\_add (ap (ap c\_2Einteger\_2Eint\_sub V0x) V1y)) V1y) = V0x))) \quad (60)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((ap (ap c\_2Einteger\_2Eint\_sub V0x) V0x) = (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0))) \quad (61)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.((ap c\_2Einteger\_2Eint\_neg (ap (ap c\_2Einteger\_2Eint\_sub V0x) V1y)) = (ap (ap c\_2Einteger\_2Eint\_sub V1y) V0x)))) \quad (62)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.(\forall V2z \in ty\_2Einteger\_2Eint.((ap (ap c\_2Einteger\_2Eint\_mul V0x) (ap (ap c\_2Einteger\_2Eint\_sub V1y) V2z)) = (ap (ap c\_2Einteger\_2Eint\_sub (ap (ap c\_2Einteger\_2Eint\_mul V0x) V1y)) (ap (ap c\_2Einteger\_2Eint\_mul V0x) V2z))))))) \quad (63)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.(\forall V2z \in ty\_2Einteger\_2Eint.((ap (ap c\_2Einteger\_2Eint\_mul (ap (ap c\_2Einteger\_2Eint\_sub V0x) V1y)) V2z) = (ap (ap c\_2Einteger\_2Eint\_sub (ap (ap c\_2Einteger\_2Eint\_mul V0x) V2z)) (ap (ap c\_2Einteger\_2Eint\_mul V1y) V2z))))))) \quad (64)$$



Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (((ap (ap c\_2Einteger\_2Eint\_mul V0x) V1y) = (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) \Leftrightarrow ((V0x = (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) \vee \\
& (V1y = (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0))))))
\end{aligned} \tag{65}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\
& ((ap c\_2Einteger\_2Eint\_of\_num V0m) = (ap c\_2Einteger\_2Eint\_of\_num \\
& V1n)) \Leftrightarrow (V0m = V1n))))
\end{aligned} \tag{66}$$

Assume the following.

$$\begin{aligned}
& (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\
& (ap (ap c\_2Einteger\_2Eint\_mul (ap c\_2Einteger\_2Eint\_of\_num \\
& V0m)) (ap c\_2Einteger\_2Eint\_of\_num V1n)) = (ap c\_2Einteger\_2Eint\_of\_num \\
& (ap (ap c\_2Earithmic\_2E\_2A V0m) V1n))))))
\end{aligned} \tag{67}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty\_2Einteger\_2Eint. (\forall V1b \in ty\_2Einteger\_2Eint. \\
& (\forall V2c \in ty\_2Einteger\_2Eint. (\forall V3d \in ty\_2Einteger\_2Eint. \\
& (((ap (ap c\_2Einteger\_2Eint\_sub (ap (ap c\_2Einteger\_2Eint\_add \\
& V0a) V1b)) (ap (ap c\_2Einteger\_2Eint\_add V2c) V3d)) = (ap (ap c\_2Einteger\_2Eint\_add \\
& (ap (ap c\_2Einteger\_2Eint\_sub V0a) V2c)) (ap (ap c\_2Einteger\_2Eint\_sub \\
& V1b) V3d)))))))))
\end{aligned} \tag{68}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2z \in ty\_2Einteger\_2Eint. ((V0x = (ap (ap c\_2Einteger\_2Eint\_sub \\
& V1y) V2z)) \Leftrightarrow ((ap (ap c\_2Einteger\_2Eint\_add V0x) V2z) = V1y))))))
\end{aligned} \tag{69}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2z \in ty\_2Einteger\_2Eint. (((ap (ap c\_2Einteger\_2Eint\_sub \\
& V0x) V1y) = V2z) \Leftrightarrow (V0x = (ap (ap c\_2Einteger\_2Eint\_add V2z) V1y))))))
\end{aligned} \tag{70}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& (\forall V2z \in ty\_2Einteger\_2Eint. (((p (ap (ap c\_2Einteger\_2Eint\_divides \\
& V0x) V1y)) \wedge (p (ap (ap c\_2Einteger\_2Eint\_divides V1y) V2z))) \Rightarrow \\
& (p (ap (ap c\_2Einteger\_2Eint\_divides V0x) V2z))))))
\end{aligned} \tag{71}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1q \in ty\_2Einteger\_2Eint. \\
& ((p (ap (ap c\_2Einteger\_2Eint\_divides V0p) (ap (ap c\_2Einteger\_2Eint\_mul \\
& V0p) V1q))) \wedge (p (ap (ap c\_2Einteger\_2Eint\_divides V0p) (ap (ap \\
& c\_2Einteger\_2Eint\_mul V1q) V0p))))))
\end{aligned} \tag{72}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1q \in ty\_2Einteger\_2Eint. \\
& (\forall V2r \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_divides \\
& V0p) V1q)) \Rightarrow (p (ap (ap c\_2Einteger\_2Eint\_divides V0p) (ap (ap c\_2Einteger\_2Eint\_mul \\
& V1q) V2r))))))
\end{aligned} \tag{73}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1q \in ty\_2Einteger\_2Eint. \\
& (\forall V2r \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_divides \\
& V0p) V1q)) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_divides V0p) (ap (ap \\
& c\_2Einteger\_2Eint\_add V1q) V2r))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_divides \\
& V0p) V2r))))))
\end{aligned} \tag{74}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1q \in ty\_2Einteger\_2Eint. \\
& (((p (ap (ap c\_2Einteger\_2Eint\_divides V0p) (ap c\_2Einteger\_2Eint\_neg \\
& V1q))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_divides V0p) V1q))) \wedge ((p \\
& (ap (ap c\_2Einteger\_2Eint\_divides (ap c\_2Einteger\_2Eint\_neg \\
& V0p)) V1q)) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_divides V0p) V1q))))))
\end{aligned} \tag{75}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1q \in ty\_2Einteger\_2Eint. \\
& (\forall V2r \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_divides \\
& V0p) V1q)) \Rightarrow ((p (ap (ap c\_2Einteger\_2Eint\_divides V0p) (ap (ap \\
& c\_2Einteger\_2Eint\_sub V1q) V2r))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_divides \\
& V0p) V2r))))))
\end{aligned} \tag{76}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
& ((ap (ap c\_2Einteger\_2Eint\_sub V0x) V1y) = (ap (ap c\_2Einteger\_2Eint\_add \\
& V0x) (ap c\_2Einteger\_2Eint\_neg V1y))))))
\end{aligned} \tag{77}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \tag{78}$$

Assume the following.

$$(\forall V0A \in 2.((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (79)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (80)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (81)$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (82)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (p V1q) \Leftrightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg(p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \quad (83)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (p V1q) \wedge (p V2r)) \Leftrightarrow (((p V0p) \vee ((\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p)))))))) \quad (84)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (p V1q) \vee (p V2r)) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))) \quad (85)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (p V1q) \Rightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((\neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))) \quad (86)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p)))))) \quad (87)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p)))) \quad (88)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (\neg(p V1q)))))) \quad (89)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V0p)))))) \quad (90)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p V0p) \vee (p V1q))) \Rightarrow (\neg(p V1q)))))) \quad (91)$$

Assume the following.

$$(\forall V0p \in 2. ((\neg(\neg(p V0p))) \Rightarrow (p V0p))) \quad (92)$$

**Theorem 1**

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum. (\forall V1n \in ty\_2Enum\_2Enum. ( \\ & \quad \forall V2a \in ty\_2Enum\_2Enum. (\forall V3b \in ty\_2Einteger\_2Eint. \\ & \quad \forall V4u \in ty\_2Enum\_2Enum. (\forall V5v \in ty\_2Einteger\_2Eint. \\ & \quad \forall V6p \in ty\_2Einteger\_2Eint. (\forall V7q \in ty\_2Einteger\_2Eint. \\ & \quad \forall V8x \in ty\_2Einteger\_2Eint. (\forall V9d \in ty\_2Enum\_2Enum. \\ & \quad (((V9d = (ap (ap c\_2Egcd\_2Egcd (ap (ap c\_2Earithmetic\_2E\_2A V4u) \\ & V0m)) (ap (ap c\_2Earithmetic\_2E\_2A V2a) V1n))) \wedge (((ap c\_2Einteger\_2Eint\_of\_num \\ & \quad V9d) = (ap (ap c\_2Einteger\_2Eint\_add (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad (ap (ap c\_2Einteger\_2Eint\_mul V6p) (ap c\_2Einteger\_2Eint\_of\_num \\ & \quad V4u))) (ap c\_2Einteger\_2Eint\_of\_num V0m))) (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad (ap (ap c\_2Einteger\_2Eint\_mul V7q) (ap c\_2Einteger\_2Eint\_of\_num \\ & \quad V2a))) (ap c\_2Einteger\_2Eint\_of\_num V1n)))))) \wedge ((\neg(V0m = c\_2Enum\_2E0)) \wedge \\ & ((\neg(V1n = c\_2Enum\_2E0)) \wedge ((\neg(V2a = c\_2Enum\_2E0)) \wedge (\neg(V4u = c\_2Enum\_2E0)))))) \Rightarrow \\ & (((p (ap (ap c\_2Einteger\_2Eint\_divides (ap c\_2Einteger\_2Eint\_of\_num \\ & \quad V0m)) (ap (ap c\_2Einteger\_2Eint\_add (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad (ap c\_2Einteger\_2Eint\_of\_num V2a) V8x)) V3b))) \wedge (p (ap (ap c\_2Einteger\_2Eint\_divides \\ & \quad (ap c\_2Einteger\_2Eint\_of\_num V1n)) (ap (ap c\_2Einteger\_2Eint\_add \\ & \quad (ap (ap c\_2Einteger\_2Eint\_mul (ap c\_2Einteger\_2Eint\_of\_num \\ & \quad V4u) V8x)) V5v)))) \Leftrightarrow ((p (ap (ap c\_2Einteger\_2Eint\_divides (ap \\ & \quad (ap c\_2Einteger\_2Eint\_mul (ap c\_2Einteger\_2Eint\_of\_num V0m)) \\ & \quad (ap c\_2Einteger\_2Eint\_of\_num V1n))) (ap (ap c\_2Einteger\_2Eint\_add \\ & \quad (ap (ap c\_2Einteger\_2Eint\_add (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad (ap c\_2Einteger\_2Eint\_of\_num V9d) V8x)) (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad (ap (ap c\_2Einteger\_2Eint\_mul V5v) (ap c\_2Einteger\_2Eint\_of\_num \\ & \quad V0m))) V6p))) (ap (ap c\_2Einteger\_2Eint\_mul (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad V3b) (ap c\_2Einteger\_2Eint\_of\_num V1n))) V7q)))) \wedge (p (ap (ap \\ & \quad c\_2Einteger\_2Eint\_divides (ap c\_2Einteger\_2Eint\_of\_num \\ & \quad V9d) (ap (ap c\_2Einteger\_2Eint\_sub (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad (ap c\_2Einteger\_2Eint\_of\_num V2a) V5v)) (ap (ap c\_2Einteger\_2Eint\_mul \\ & \quad (ap c\_2Einteger\_2Eint\_of\_num V4u) V3b)))))))))))))) \end{aligned}$$