

thm\_2Eint\_arith\_2Ein\_subtractive\_range  
 (TMGM-  
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October 26, 2020

**Definition 1** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p (ap P x)) \text{ then } (\lambda x. x \in A \wedge p$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A. \lambda a : \iota. (\lambda V0P \in (2^{A\_27a}). (ap V0P (ap (c\_2Emin\_2E\_40 A$

**Definition 4** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 5** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2. V0x)) (\lambda V1x \in 2. V1x))$

**Definition 6** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A. \lambda a : \iota. (\lambda V0P \in (2^{A\_27a}). (ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})$

**Definition 7** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2. V0t))$ .

**Definition 8** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2. (ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2EF))$

**Definition 9** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (1)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$\text{nonempty } ty\_2Enum\_2Enum \quad (2)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^\omega) \quad (3)$$

**Definition 10** We define  $c\_2Enum\_2E0$  to be  $(ap c\_2Enum\_2EABS\_num c\_2Enum\_2EZERO\_REP)$ .

Let  $ty\_2Einteger\_2Eint : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \quad (4)$$

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (5)$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (6)$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2Eint}) \quad (7)$$

**Definition 11** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (c\_2Emin\_2E40\ (t$

Let  $c\_2Einteger\_2Etint\_lt : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)}) \quad (8)$$

**Definition 12** We define  $c\_2Einteger\_2Eint\_lt$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$

Let  $c\_2Einteger\_2Etint\_add : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)} \quad (9)$$

Let  $c\_2Einteger\_2Etint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)}) \quad (10)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint)^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}} \quad (11)$$

**Definition 13** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)$

**Definition 14** We define  $c\_2Einteger\_2Eint\_add$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger$

Let  $c\_2Einteger\_2Etint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_neg \in ((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) \quad (12)$$

**Definition 15** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap\ c\_2Einteger\_2Eint\_neg\ V0T1)$

**Definition 16** We define  $c\_2Einteger\_2Eint\_le$  to be  $\lambda V0x \in ty\_2Einteger\_2Eint.\lambda V1y \in ty\_2Einteger\_2Eint.(ap\ c\_2Einteger\_2Eint\_le\ V0x\ V1y)$

**Definition 17** We define  $c\_2Einteger\_2Eint\_sub$  to be  $\lambda V0x \in ty\_2Einteger\_2Eint.\lambda V1y \in ty\_2Einteger\_2Eint.(ap\ c\_2Einteger\_2Eint\_sub\ V0x\ V1y)$

Assume the following.

$$True \quad (13)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (14)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \quad (15)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (16)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg(p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p\ V0t))))))) \quad (17)$$

Assume the following.

$$(\forall V0y \in ty\_2Einteger\_2Eint.(\forall V1x \in ty\_2Einteger\_2Eint.((ap\ (ap\ c\_2Einteger\_2Eint\_add\ V1x)\ V0y) = (ap\ (ap\ c\_2Einteger\_2Eint\_add\ V0y)\ V1x)))))) \quad (18)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((ap\ (ap\ c\_2Einteger\_2Eint\_add\ (ap\ c\_2Einteger\_2Eint\_of\_num\ c\_2Enum\_2E0))\ V0x) = V0x))) \quad (19)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.(\forall V2z \in ty\_2Einteger\_2Eint.((p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le\ (ap\ (ap\ c\_2Einteger\_2Eint\_add\ V0x)\ V1y))\ (ap\ (ap\ c\_2Einteger\_2Eint\_add\ V0x)\ V2z))) \Leftrightarrow (p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le\ V1y)\ V2z))))))) \quad (20)$$

Assume the following.

$$\begin{aligned}
 & (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
 & ((p (ap (ap c\_2Einteger\_2Eint\_lt V0x)) (ap (ap c\_2Einteger\_2Eint\_add \\
 & V0x) V1y))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
 & c\_2Enum\_2E0)) V1y)))))) \\
 \end{aligned} \tag{21}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
 & (\forall V2z \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_lt \\
 & (ap (ap c\_2Einteger\_2Eint\_sub V0x) V1y)) V2z)) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_lt \\
 & V0x) (ap (ap c\_2Einteger\_2Eint\_add V2z) V1y))))))) \\
 \end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
 & (\forall V2z \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_lt \\
 & (ap (ap c\_2Einteger\_2Eint\_sub V0x) V1y)) V2z)) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_lt \\
 & (ap (ap c\_2Einteger\_2Eint\_add V0x) V2z)) V1y))))))) \\
 \end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
 & (\forall V2z \in ty\_2Einteger\_2Eint. ((p (ap (ap c\_2Einteger\_2Eint\_le \\
 & (ap (ap c\_2Einteger\_2Eint\_sub V0x) V1y)) V2z)) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_le \\
 & V0x) (ap (ap c\_2Einteger\_2Eint\_add V2z) V1y))))))) \\
 \end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
 & ((p (ap (ap c\_2Einteger\_2Eint\_le (ap c\_2Einteger\_2Eint\_neg \\
 & V0x)) (ap c\_2Einteger\_2Eint\_neg V1y))) \Leftrightarrow (p (ap (ap c\_2Einteger\_2Eint\_le \\
 & V1y) V0x)))))) \\
 \end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned}
 & (\forall V0x \in ty\_2Einteger\_2Eint. (\forall V1y \in ty\_2Einteger\_2Eint. \\
 & ((ap (ap c\_2Einteger\_2Eint\_sub V0x) (ap (ap c\_2Einteger\_2Eint\_sub \\
 & V0x) V1y)) = V1y))) \\
 \end{aligned} \tag{26}$$

### Theorem 1

$$\begin{aligned}
 & (\forall Vhigh \in ty\_2Einteger\_2Eint. (\forall V1d \in ty\_2Einteger\_2Eint. \\
 & (\forall V2x \in ty\_2Einteger\_2Eint. (((p (ap (ap c\_2Einteger\_2Eint\_le \\
 & (ap (ap c\_2Einteger\_2Eint\_sub V0high) V1d)) V2x)) \wedge (p (ap (ap c\_2Einteger\_2Eint\_lt \\
 & V2x) V0high))) \Leftrightarrow (\exists V3j \in ty\_2Einteger\_2Eint. ((V2x = (ap \\
 & (ap c\_2Einteger\_2Eint\_sub V0high) V3j)) \wedge ((p (ap (ap c\_2Einteger\_2Eint\_lt \\
 & (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) V3j)) \wedge (p (ap (ap \\
 & c\_2Einteger\_2Eint\_le V3j) V1d))))))) \\
 \end{aligned}$$