

thm\_2Eint\_arith\_2Epositive\_product\_implication  
 (TMcafGkppTi3YFUnU4ZbMuHq36xS9E5p8D5)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A. \lambda a : \iota. (\lambda V0P \in (2^{A\_27a}). (ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})) (\lambda V1P \in 2.V1P)) (\lambda V2P \in 2.V2P)))$

**Definition 4** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p \ P \Rightarrow p \ Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2. (ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2EF))$

**Definition 7** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2. (ap (c\_2Ebool\_2E\_7E V2t) c\_2Ebool\_2EF))))$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty \ ty\_2Enum\_2Enum \quad (1)$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A0. & nonempty \ A0 \Rightarrow \forall A1. & nonempty \ A1 \Rightarrow & nonempty \ (ty\_2Epair\_2Eprod \\ & A0 \ A1) \end{aligned} \quad (2)$$

Let  $ty\_2Einteger\_2Eint : \iota$  be given. Assume the following.

$$nonempty \ ty\_2Einteger\_2Eint \quad (3)$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod \ ty\_2Enum\_2Enum \ ty\_2Enum\_2Enum)}) \ ty\_2Einteger\_2Eint \ REP\_CLASS) \quad (4)$$

**Definition 8** We define  $c_2Emin_2E40$  to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. p (ap P x)) \text{ then } (\lambda x. x \in A \wedge p$  of type  $\iota \Rightarrow \iota$ .

**Definition 9** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (c\_2Emin\_2E\_40\ (ty$

Let  $c_2Einteger_2Etint\_mul : \iota$  be given. Assume the following.

$$c\_Einteger\_2Etint\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\\ ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)} \quad (5)$$

Let  $c_2Einteger_2Etint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)})\^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)} \quad (6)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})$$

**Definition 10** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum)$

**Definition 11** We define  $c\_2Einteger\_2Eint\_mul$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint. \lambda V1T2 \in ty\_2Einteger\_2Eint.$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ZERO\_REP \in \omega \quad (8)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{omega}) \quad (9)$$

**Definition 12** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

Let  $c_2Einteger_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (10)$$

Let  $c_2Einteger_2Etint\_lt : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}) (ty\_2Epair\_2Eprod\ ty\_2Enum\\ (11)$$

**Definition 13** We define  $c\_2Einteger\_2Eint\_lt$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint. \lambda V1T2 \in ty\_2Einteger$

**Definition 14** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in$

Assume the following.

*True* (12)

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (13)$$

Assume the following.

$$(\forall V0t \in 2.(((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee (p V0t)) \Leftrightarrow (p V0t)))))) \quad (14)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (15)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (16)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (17)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (18)$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x\_27 \in 2.(\forall V2y \in 2.(\forall V3y\_27 \in 2.(((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27)))))) \quad (19)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in ty\_2Einteger\_2Eint. (\forall V1q \in ty\_2Einteger\_2Eint. \\
& (((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_mul V0p) V1q))) \Leftrightarrow ((( \\
& p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V0p)) \wedge (p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V1q))) \vee ((p (ap (ap c\_2Einteger\_2Eint\_lt V0p) ( \\
& ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0))) \wedge (p (ap (ap c\_2Einteger\_2Eint\_lt \\
& V1q) (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)))))) \wedge ((p ( \\
& ap (ap c\_2Einteger\_2Eint\_lt (ap (ap c\_2Einteger\_2Eint\_mul V0p) \\
& V1q)) (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0))) \Leftrightarrow (((p (ap \\
& (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num c\_2Enum\_2E0)) \\
& V0p)) \wedge (p (ap (ap c\_2Einteger\_2Eint\_lt V1q) (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)))) \vee ((p (ap (ap c\_2Einteger\_2Eint\_lt V0p) (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0))) \wedge (p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V1q)))))))
\end{aligned} \tag{20}$$

### Theorem 1

$$\begin{aligned}
& (\forall V0c \in ty\_2Einteger\_2Eint. (\forall V1d \in ty\_2Einteger\_2Eint. \\
& (((p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V0c)) \wedge (p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) V1d))) \Rightarrow (p (ap (ap c\_2Einteger\_2Eint\_lt (ap c\_2Einteger\_2Eint\_of\_num \\
& c\_2Enum\_2E0)) (ap (ap c\_2Einteger\_2Eint\_mul V0c) V1d))))))
\end{aligned}$$