

thm_2Einteger_2EINT__LTE__TRANS (TMcrsvGP6o59irrWmSpfduWEZV6VJemm8oq)

October 26, 2020

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_ET$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 4 We define $c_2Ebool_2E_EF$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 6 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_EF$

Let $ty_2Eenum_2Eenum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eenum_2Eenum \tag{1}$$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \tag{2}$$

Let $c_2Einteger_2E_etint_eq : \iota$ be given. Assume the following.

$$c_2Einteger_2E_etint_eq \in ((2^{(ty_2Epair_2Eprod\ ty_2Eenum_2Eenum\ ty_2Eenum_2Eenum)})(ty_2Epair_2Eprod\ ty_2Eenum_2Eenum) \tag{3}$$

Let $ty_2Einteger_2E_eint : \iota$ be given. Assume the following.

$$nonempty\ ty_2Einteger_2E_eint \tag{4}$$

Let $c_2Einteger_2E_eint_ABS_CLASS : \iota$ be given. Assume the following.

$$c_2Einteger_2E_eint_ABS_CLASS \in (ty_2Einteger_2E_eint^{(2^{(ty_2Epair_2Eprod\ ty_2Eenum_2Eenum\ ty_2Eenum_2Eenum)})} \tag{5}$$

Definition 7 We define $c_2Einteger_2Eint_ABS$ to be $\lambda V0r \in (ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)$.
Let $c_2Einteger_2Eint_REP_CLASS : \iota$ be given. Assume the following.

$$c_2Einteger_2Eint_REP_CLASS \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})_{ty_2Einteger_2Eint_REP_CLASS})$$

(6)

Definition 8 We define c_2Emin_2E40 to be $\lambda A.\lambda P \in 2^A$. **if** $(\exists x \in A.p (ap\ P\ x))$ **then** (the $(\lambda x.x \in A \wedge p\ x)$ of type $\iota \Rightarrow \iota$).

Definition 9 We define $c_2Einteger_2Eint_REP$ to be $\lambda V0a \in ty_2Einteger_2Eint$. $(ap\ (c_2Emin_2E40\ (ty_2Einteger_2Eint_REP\ a)))$.
Let $c_2Einteger_2Eint_lt : \iota$ be given. Assume the following.

$$c_2Einteger_2Eint_lt \in ((2^{(ty_2Epair_2Eprod\ ty_2Enum_2Enum\ ty_2Enum_2Enum)})_{(ty_2Epair_2Eprod\ ty_2Enum_2Enum)})$$

(7)

Definition 10 We define $c_2Einteger_2Eint_lt$ to be $\lambda V0T1 \in ty_2Einteger_2Eint$. $\lambda V1T2 \in ty_2Einteger_2Eint$.

Definition 11 We define $c_2Ebool_2E5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E21\ 2)\ (\lambda V2t \in 2))))$.

Definition 12 We define $c_2Einteger_2Eint_le$ to be $\lambda V0x \in ty_2Einteger_2Eint$. $\lambda V1y \in ty_2Einteger_2Eint$.

Definition 13 We define $c_2Ecombin_2EK$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(\lambda V0x \in A_27a.(\lambda V1y \in A_27b.V0x))$.

Definition 14 We define $c_2Ecombin_2ES$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.(\lambda V0f \in ((A_27c^{A_27b})^{A_27a}))$.

Definition 15 We define $c_2Ecombin_2EI$ to be $\lambda A_27a : \iota.(ap\ (ap\ (c_2Ecombin_2ES\ A_27a\ (A_27a^{A_27a}))\ A_27a))$.

Definition 16 We define $c_2Equotient_2E2D_2D_3E$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda A_27d : \iota.\lambda V0f \in ((A_27c^{A_27b})^{A_27a})$.

Definition 17 We define $c_2Equotient_2E3D_3D_3E$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0R1 \in ((2^{A_27a})^{A_27b})$.

Definition 18 We define $c_2Ebool_2E2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_2Ebool_2E21\ 2)\ (\lambda V2t \in 2))))$.

Definition 19 We define $c_2Equotient_2EQUOTIENT$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0R \in ((2^{A_27a})^{A_27b})$.

Definition 20 We define $c_2Ecombin_2EW$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(\lambda V0f \in ((A_27b^{A_27a})^{A_27a})).(\lambda V1x \in A_27a)$.

Definition 21 We define $c_2Equotient_2Erespects$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(c_2Ecombin_2EW\ A_27a\ A_27b)$.

Definition 22 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a})).(ap\ V1f\ V0x))$.

Definition 23 We define $c_2Ebool_2ERES_FORALL$ to be $\lambda A_27a : \iota.(\lambda V0p \in (2^{A_27a})).(\lambda V1m \in (2^{A_27a})).$

Definition 24 We define $c_2Equotient_2EEQUIV$ to be $\lambda A_27a : \iota.\lambda V0E \in ((2^{A_27a})^{A_27a})).(ap\ (c_2Ebool_2EIN\ E))$.

Assume the following.

$$True \quad (8)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((\\ & (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \end{aligned} \quad (9)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in \\ & A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \end{aligned} \quad (10)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p V0A) \wedge (\\ & (p V1B) \vee (p V2C)) \Leftrightarrow (((p V0A) \wedge (p V1B)) \vee ((p V0A) \wedge (p V2C)))))) \end{aligned} \quad (11)$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow \\ & ((p V1t2) \Rightarrow (p V2t3)) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \end{aligned} \quad (12)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a. ((ap \ (c_2Ecombin_2El \\ & A_27a) \ V0x) = V0x)) \end{aligned} \quad (13)$$

Assume the following.

$$\begin{aligned} & (\forall V0p \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (\forall V1q \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & ((p \ (ap \ (ap \ c_2Einteger_2Etint_eq \ V0p) \ V1q)) \Leftrightarrow ((ap \ c_2Einteger_2Etint_eq \\ & \ V0p) = (ap \ c_2Einteger_2Etint_eq \ V1q)))) \end{aligned} \quad (14)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (\forall V1y \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (\forall V2z \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (((p \ (ap \ (ap \ c_2Einteger_2Etint_lt \ V0x) \ V1y)) \wedge (p \ (ap \ (ap \ c_2Einteger_2Etint_lt \\ & \ V1y) \ V2z))) \Rightarrow (p \ (ap \ (ap \ c_2Einteger_2Etint_lt \ V0x) \ V2z)))))) \end{aligned} \quad (15)$$

Assume the following.

$$\begin{aligned} & (\forall V0x1 \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (\forall V1x2 \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (\forall V2y1 \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (\forall V3y2 \in (ty_2Epair_2Eprod \ ty_2Enum_2Enum \ ty_2Enum_2Enum). \\ & (((p \ (ap \ (ap \ c_2Einteger_2Etint_eq \ V0x1) \ V1x2)) \wedge (p \ (ap \ (ap \ c_2Einteger_2Etint_eq \\ & \ V2y1) \ V3y2))) \Rightarrow ((p \ (ap \ (ap \ c_2Einteger_2Etint_lt \ V0x1) \ V2y1)) \Leftrightarrow \\ & (p \ (ap \ (ap \ c_2Einteger_2Etint_lt \ V1x2) \ V3y2)))))) \end{aligned} \quad (16)$$

Assume the following.

$$(p (ap (ap (ap (ap (c_2Equotient_2EQUOTIENT (ty_2Epair_2Eprod ty_2Enum_2Enum ty_2Enum_2Enum) ty_2Einteger_2Eint) c_2Einteger_2Etint_eq) c_2Einteger_2Eint_ABS) c_2Einteger_2Eint_REP))) (17)$$

Assume the following.

$$(\forall V0x \in ty_2Einteger_2Eint. (\forall V1y \in ty_2Einteger_2Eint. ((p (ap (ap c_2Einteger_2Eint_le V0x) V1y)) \Leftrightarrow ((p (ap (ap c_2Einteger_2Eint_lt V0x) V1y)) \vee (V0x = V1y)))))) (18)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (p (ap (ap (ap (c_2Equotient_2EQUOTIENT A_27a A_27a) (c_2Emin_2E_3D A_27a)) (c_2Ecombin_2EI A_27a)) (c_2Ecombin_2EI A_27a))) (19)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow \forall A_27c. \\ & \quad nonempty A_27c \Rightarrow \forall A_27d.nonempty A_27d \Rightarrow (\forall V0R1 \in (\\ & \quad (2^{A_27a})^{A_27a}). (\forall V1abs1 \in (A_27c^{A_27a}). (\forall V2rep1 \in \\ & \quad (A_27a^{A_27c}). ((p (ap (ap (ap (c_2Equotient_2EQUOTIENT A_27a A_27c) \\ & \quad V0R1) V1abs1) V2rep1)) \Rightarrow (\forall V3R2 \in ((2^{A_27b})^{A_27b}). (\forall V4abs2 \in \\ & \quad (A_27d^{A_27b}). (\forall V5rep2 \in (A_27b^{A_27d}). ((p (ap (ap (ap (c_2Equotient_2EQUOTIENT \\ & \quad A_27b A_27d) V3R2) V4abs2) V5rep2)) \Rightarrow (p (ap (ap (ap (c_2Equotient_2EQUOTIENT \\ & \quad (A_27b^{A_27a}) (A_27d^{A_27c})) (ap (ap (c_2Equotient_2E_3D_3D_3D_3E \\ & \quad A_27a A_27b) V0R1) V3R2)) (ap (ap (c_2Equotient_2E_2D_2D_3E A_27c \\ & \quad A_27b A_27a A_27d) V2rep1) V4abs2)) (ap (ap (c_2Equotient_2E_2D_2D_3E \\ & \quad A_27a A_27d A_27c A_27b) V1abs1) V5rep2)))))))))) (20) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow \forall A_27c. \\ & \quad nonempty A_27c \Rightarrow \forall A_27d.nonempty A_27d \Rightarrow (\forall V0R1 \in (\\ & \quad (2^{A_27a})^{A_27a}). (\forall V1abs1 \in (A_27c^{A_27a}). (\forall V2rep1 \in \\ & \quad (A_27a^{A_27c}). ((p (ap (ap (ap (c_2Equotient_2EQUOTIENT A_27a A_27c) \\ & \quad V0R1) V1abs1) V2rep1)) \Rightarrow (\forall V3R2 \in ((2^{A_27b})^{A_27b}). (\forall V4abs2 \in \\ & \quad (A_27d^{A_27b}). (\forall V5rep2 \in (A_27b^{A_27d}). ((p (ap (ap (ap (c_2Equotient_2EQUOTIENT \\ & \quad A_27b A_27d) V3R2) V4abs2) V5rep2)) \Rightarrow (\forall V6f \in (A_27d^{A_27c}). \\ & \quad ((\lambda V7x \in A_27c.(ap V6f V7x)) = (ap (ap (ap (c_2Equotient_2E_2D_2D_3E \\ & \quad A_27c A_27b A_27a A_27d) V2rep1) V4abs2) (\lambda V8x \in A_27a.(ap V5rep2 \\ & \quad (ap V6f (ap V1abs1 V8x)))))))))))))) (21) \end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0REL \in ((2^{A_27a})^{A_27a}).(\forall V1abs \in (A_27b^{A_27a}). \\
& \quad (\forall V2rep \in (A_27a^{A_27b}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27a\ A_27b)\ V0REL)\ V1abs)\ V2rep))) \Rightarrow (\forall V3x1 \in A_27a.(\forall V4x2 \in \\
& \quad A_27a.((p\ (ap\ (ap\ V0REL\ V3x1)\ V4x2)) \Rightarrow (p\ (ap\ (ap\ V0REL\ V3x1)\ (ap\ V2rep \\
& \quad (ap\ V1abs\ V4x2))))))))))
\end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0R \in ((2^{A_27a})^{A_27a}).(\forall V1abs \in (A_27b^{A_27a}). \\
& \quad (\forall V2rep \in (A_27a^{A_27b}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27a\ A_27b)\ V0R)\ V1abs)\ V2rep))) \Rightarrow (\forall V3f \in (2^{A_27b}).((p\ (\\
& \quad ap\ (c_2Ebool_2E_21\ A_27b)\ V3f)) \Leftrightarrow (p\ (ap\ (ap\ (c_2Ebool_2ERES_FORALL \\
& \quad A_27a)\ (ap\ (c_2Equotient_2ERespects\ A_27a\ 2)\ V0R))\ (ap\ (ap\ (ap \\
& \quad (c_2Equotient_2E_2D_2D_3E\ A_27a\ 2\ A_27b\ 2)\ V1abs)\ (c_2Ecombin_2EI \\
& \quad 2))\ V3f))))))))))
\end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\
& \quad \forall V0R \in ((2^{A_27a})^{A_27a}).(\forall V1abs \in (A_27b^{A_27a}). \\
& \quad (\forall V2rep \in (A_27a^{A_27b}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27a\ A_27b)\ V0R)\ V1abs)\ V2rep))) \Rightarrow (\forall V3f \in (2^{A_27a}).(\forall V4g \in \\
& \quad (2^{A_27a}).((p\ (ap\ (ap\ (ap\ (ap\ (c_2Equotient_2E_3D_3D_3D_3E\ A_27a \\
& \quad 2)\ V0R)\ (c_2Emin_2E_3D\ 2)\ V3f)\ V4g)) \Rightarrow ((p\ (ap\ (ap\ (c_2Ebool_2ERES_FORALL \\
& \quad A_27a)\ (ap\ (c_2Equotient_2ERespects\ A_27a\ 2)\ V0R))\ V3f)) \Leftrightarrow (p\ (\\
& \quad ap\ (ap\ (c_2Ebool_2ERES_FORALL\ A_27a)\ (ap\ (c_2Equotient_2ERespects \\
& \quad A_27a\ 2)\ V0R))\ V4g))))))))))
\end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow \forall A_27c. \\
& \quad nonempty\ A_27c \Rightarrow \forall A_27d.nonempty\ A_27d \Rightarrow (\forall V0R1 \in (\\
& \quad (2^{A_27a})^{A_27a}).(\forall V1abs1 \in (A_27c^{A_27a}).(\forall V2rep1 \in \\
& \quad (A_27a^{A_27c}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT\ A_27a\ A_27c) \\
& \quad V0R1)\ V1abs1)\ V2rep1))) \Rightarrow (\forall V3R2 \in ((2^{A_27b})^{A_27b}).(\forall V4abs2 \in \\
& \quad (A_27d^{A_27b}).(\forall V5rep2 \in (A_27b^{A_27d}).((p\ (ap\ (ap\ (ap\ (c_2Equotient_2EQUOTIENT \\
& \quad A_27b\ A_27d)\ V3R2)\ V4abs2)\ V5rep2))) \Rightarrow (\forall V6f \in (A_27b^{A_27a}). \\
& \quad (\forall V7g \in (A_27b^{A_27a}).(\forall V8x \in A_27a.(\forall V9y \in \\
& \quad A_27a.(((p\ (ap\ (ap\ (ap\ (ap\ (c_2Equotient_2E_3D_3D_3D_3E\ A_27a \\
& \quad A_27b)\ V0R1)\ V3R2)\ V6f)\ V7g)) \wedge (p\ (ap\ (ap\ V0R1\ V8x)\ V9y))) \Rightarrow (p\ (ap\ (\\
& \quad ap\ V3R2\ (ap\ V6f\ V8x))\ (ap\ V7g\ V9y))))))))))))))
\end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0E \in ((2^{A_{27a}})^{A_{27a}}). \\
& (\forall V1P \in (2^{A_{27a}}). ((p (ap (c_2Equotient_2EEQUIV A_{27a}) \\
V0E)) \Rightarrow ((p (ap (ap (c_2Ebool_2ERES_FORALL A_{27a}) (ap (c_2Equotient_2Erespects \\
A_{27a} 2) V0E)) V1P)) \Leftrightarrow (p (ap (c_2Ebool_2E.21 A_{27a}) V1P))))))
\end{aligned} \tag{26}$$

Theorem 1

$$\begin{aligned}
& (\forall V0x \in ty_2Einteger_2Eint. (\forall V1y \in ty_2Einteger_2Eint. \\
& (\forall V2z \in ty_2Einteger_2Eint. (((p (ap (ap c_2Einteger_2Eint_lt \\
V0x) V1y)) \wedge (p (ap (ap c_2Einteger_2Eint_le V1y) V2z))) \Rightarrow (p (ap \\
(ap c_2Einteger_2Eint_lt V0x) V2z))))))
\end{aligned}$$