# thm_2Einteger_2EINT__SUMSQ (TMbTZr8ganKNiGCD7cpa773QNGMPoRsEjaq)

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**Definition 1** *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$
*of type* $\iota \Rightarrow \iota$.

**Definition 2** *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

**Definition 3** *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

**Definition 4** *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

**Definition 5** *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$
*of type* $\iota$.

**Definition 6** *We define* c_2Ebool_2E_5C_2F *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{1}$$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod$$
$$A0\ A1) \tag{2}$$

Let $ty\_2Einteger\_2Eint : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \tag{3}$$

Let $c\_2Einteger\_2Eint\_\_REP\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_REP\_\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2}$$
$$\tag{4}$$

**Definition 7** *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$
*of type* $\iota \Rightarrow \iota$.

**Definition 8** *We define* c_2Einteger_2Eint__REP *to be* $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (\mathsf{c\_2Emin\_2E\_40}\ (ty$

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Let $c\_2Einteger\_2Etint\_\_add : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_add \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum$$
$$ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ t}$$

(5)

Let $c\_2Einteger\_2Etint\_\_eq : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_eq \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum}$$

(6)

Let $c\_2Einteger\_2Eint\_\_ABS\_\_CLASS : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_\_ABS\_\_CLASS \in (ty\_2Einteger\_2Eint^{(2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)}}}$$

(7)

**Definition 9** *We define* c\_2Einteger\_2Eint\_\_ABS *to be* $\lambda V0r \in (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2$

**Definition 10** *We define* c\_2Einteger\_2Eint\_\_add *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteg$

Let $c\_2Einteger\_2Etint\_\_lt : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2}$$

(8)

**Definition 11** *We define* c\_2Einteger\_2Eint\_\_lt *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_$

**Definition 12** *We define* c\_2Ebool\_2E\_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2E}$

**Definition 13** *We define* c\_2Einteger\_2Eint\_\_le *to be* $\lambda V0x \in ty\_2Einteger\_2Eint.\lambda V1y \in ty\_2Einteger\_2E$

**Definition 14** *We define* c\_2Ebool\_2E\_2F\_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in$

Let $c\_2Einteger\_2Etint\_\_mul : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Etint\_\_mul \in (((ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum$$
$$ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty}$$

(9)

**Definition 15** *We define* c\_2Einteger\_2Eint\_\_mul *to be* $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteg$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \tag{10}$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \tag{11}$$

**Definition 16** *We define* c\_2Enum\_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Let $c\_2Einteger\_2Eint\_of\_\_num : \iota$ be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \tag{12}$$

Assume the following.

$$True \tag{13}$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1)\Rightarrow(p\ V1t2))\Rightarrow(((p \\ V1t2)\Rightarrow(p\ V0t1))\Rightarrow((p\ V0t1)\Leftrightarrow(p\ V1t2)))))) \tag{14}$$

Assume the following.

$$(\forall V0t \in 2.(False\Rightarrow(p\ V0t))) \tag{15}$$

Assume the following.

$$(\forall V0t \in 2.(((True\wedge(p\ V0t))\Leftrightarrow(p\ V0t))\wedge(((( p\ V0t)\wedge True)\Leftrightarrow \\ (p\ V0t))\wedge(((False\wedge(p\ V0t))\Leftrightarrow False)\wedge((((p\ V0t)\wedge False)\Leftrightarrow False)\wedge \\ (((p\ V0t)\wedge(p\ V0t))\Leftrightarrow(p\ V0t))))))) \tag{16}$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p\ V0t)))\Leftrightarrow(p\ V0t)))\wedge(((\neg True)\Leftrightarrow False)\wedge \\ ((\neg False)\Leftrightarrow True))) \tag{17}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a\Rightarrow(\forall V0x \in A\_27a.((V0x = V0x)\Leftrightarrow \\ True)) \tag{18}$$

Assume the following.

$$(\forall V0t \in 2.(((True\Leftrightarrow(p\ V0t))\Leftrightarrow(p\ V0t))\wedge((((p\ V0t)\Leftrightarrow True)\Leftrightarrow \\ (p\ V0t))\wedge(((False\Leftrightarrow(p\ V0t))\Leftrightarrow(\neg(p\ V0t)))\wedge(((p\ V0t)\Leftrightarrow False)\Leftrightarrow(\neg( \\ p\ V0t)))))))) \tag{19}$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p\ V0A)\wedge(p\ V1B)))\Leftrightarrow((\neg( \\ p\ V0A))\vee(\neg(p\ V1B))))\wedge((\neg((p\ V0A)\vee(p\ V1B)))\Leftrightarrow((\neg(p\ V0A))\wedge(\neg(p\ V1B))))))) \tag{20}$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((ap\ (ap\ \mathsf{c\_2Einteger\_2Eint\_add} \\ V0x)\ (ap\ c\_2Einteger\_2Eint\_of\_\_num\ \mathsf{c\_2Enum\_2E0})) = V0x)) \tag{21}$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((ap\ (ap\ \mathsf{c\_2Einteger\_2Eint\_mul} \\ (ap\ c\_2Einteger\_2Eint\_of\_\_num\ \mathsf{c\_2Enum\_2E0}))\ V0x) = (ap\ c\_2Einteger\_2Eint\_of\_\_num \\ \mathsf{c\_2Enum\_2E0}))) \tag{22}$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_le}$$
$$(ap \ c\_2Einteger\_2Eint\_\_of\_\_num \ \mathsf{c\_2Enum\_2E0})) \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$V0x) \ V0x))))$$

$$(23)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.$$
$$(((p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_le} \ (ap \ c\_2Einteger\_2Eint\_\_of\_\_num$$
$$\mathsf{c\_2Enum\_2E0})) \ V0x)) \wedge (p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_lt} \ (ap \ c\_2Einteger\_2Eint\_\_of\_\_num$$
$$\mathsf{c\_2Enum\_2E0})) \ V1y))) \Rightarrow (p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_lt} \ (ap \ c\_2Einteger\_2Eint\_\_of\_\_num$$
$$\mathsf{c\_2Enum\_2E0})) \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_add} \ V0x) \ V1y)))))))$$

$$(24)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.$$
$$(((p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_lt} \ (ap \ c\_2Einteger\_2Eint\_\_of\_\_num$$
$$\mathsf{c\_2Enum\_2E0})) \ V0x)) \wedge (p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_le} \ (ap \ c\_2Einteger\_2Eint\_\_of\_\_num$$
$$\mathsf{c\_2Enum\_2E0})) \ V1y))) \Rightarrow (p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_lt} \ (ap \ c\_2Einteger\_2Eint\_\_of\_\_num$$
$$\mathsf{c\_2Enum\_2E0})) \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_add} \ V0x) \ V1y)))))))$$

$$(25)$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_lt}$$
$$(ap \ c\_2Einteger\_2Eint\_\_of\_\_num \ \mathsf{c\_2Enum\_2E0})) \ V0x)) \Rightarrow (\neg(V0x = \quad (26)$$
$$(ap \ c\_2Einteger\_2Eint\_\_of\_\_num \ \mathsf{c\_2Enum\_2E0})))))$$

Assume the following.

$$(\forall V0x \in ty\_2Einteger\_2Eint.((p \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_lt}$$
$$(ap \ c\_2Einteger\_2Eint\_\_of\_\_num \ \mathsf{c\_2Enum\_2E0})) \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$V0x) \ V0x))) \Leftrightarrow (\neg(V0x = (ap \ c\_2Einteger\_2Eint\_\_of\_\_num \ \mathsf{c\_2Enum\_2E0})))))$$

$$(27)$$

**Theorem 1**

$$(\forall V0x \in ty\_2Einteger\_2Eint.(\forall V1y \in ty\_2Einteger\_2Eint.$$
$$(((ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_add} \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_mul}$$
$$V0x) \ V0x)) \ (ap \ (ap \ \mathsf{c\_2Einteger\_2Eint\_\_mul} \ V1y) \ V1y)) = (ap \ c\_2Einteger\_2Eint\_\_of\_\_num$$
$$\mathsf{c\_2Enum\_2E0})) \Leftrightarrow ((V0x = (ap \ c\_2Einteger\_2Eint\_\_of\_\_num \ \mathsf{c\_2Enum\_2E0})) \wedge$$
$$(V1y = (ap \ c\_2Einteger\_2Eint\_\_of\_\_num \ \mathsf{c\_2Enum\_2E0}))))))$$