

thm\_2Einteger\_\_word\_2Ew2i\_\_lt\_\_0  
(TMJ1NuwpXHntvLKVWNAFKPzb1QYrJzYMMuG)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2F$

Let  $ty\_2Einteger\_2Eint : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Einteger\_2Eint \tag{1}$$

Let  $ty\_2Ebool\_2Eitself : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ebool\_2Eitself\ A0) \tag{2}$$

Let  $c\_2Einteger\_word\_2EINT\_MAX : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Einteger\_word\_2EINT\_MAX\ A\_27a \in (ty\_2Einteger\_2Eint^{(ty\_2Ebool\_2Eitself\ A\_27a)}) \tag{3}$$

Let  $c\_2Ebool\_2Ethe\_value : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ebool\_2Ethe\_value\ A\_27a \in (ty\_2Ebool\_2Eitself\ A\_27a) \tag{4}$$

Let  $c\_2Einteger\_word\_2EINT\_MIN : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Einteger\_word\_2EINT\_MIN\ A\_27a \in (ty\_2Einteger\_2Eint^{(ty\_2Ebool\_2Eitself\ A\_27a)}) \tag{5}$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (6)$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (7)$$

Let  $c\_2Einteger\_2Eint\_REP\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_REP\_CLASS \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{ty\_2Einteger\_2Eint\_REP\_CLASS}) \quad (8)$$

**Definition 7** We define  $c\_2Emin\_2E40$  to be  $\lambda A.\lambda P \in 2^A.$ if  $(\exists x \in A.p\ (ap\ P\ x))$  then (the  $(\lambda x.x \in A \wedge p\ x)$  of type  $\iota \Rightarrow \iota$ ).

**Definition 8** We define  $c\_2Einteger\_2Eint\_REP$  to be  $\lambda V0a \in ty\_2Einteger\_2Eint.(ap\ (c\_2Emin\_2E40\ (ty\_2Einteger\_2Eint\_REP\_CLASS\ a)))$

Let  $c\_2Einteger\_2Eint\_lt : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_lt \in ((2^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum)}) \quad (9)$$

**Definition 9** We define  $c\_2Einteger\_2Eint\_lt$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.\lambda V1T2 \in ty\_2Einteger\_2Eint.$

**Definition 10** We define  $c\_2Einteger\_2Eint\_le$  to be  $\lambda V0x \in ty\_2Einteger\_2Eint.\lambda V1y \in ty\_2Einteger\_2Eint.$

Let  $c\_2Einteger\_2Eint\_of\_num : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_of\_num \in (ty\_2Einteger\_2Eint^{ty\_2Enum\_2Enum}) \quad (10)$$

**Definition 11** We define  $c\_2Einteger\_2ENum$  to be  $\lambda V0i \in ty\_2Einteger\_2Eint.(ap\ (c\_2Emin\_2E40\ ty\_2Einteger\_2Eint\_of\_num\ i))$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (11)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (12)$$

**Definition 12** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

**Definition 13** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (13)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (14)$$

**Definition 14** We define  $c\_Enum\_ESUC$  to be  $\lambda V0m \in ty\_Enum\_Enum.(ap\ c\_Enum\_EABS\_num$

Let  $c\_Earithmic\_E\_2B : \iota$  be given. Assume the following.

$$c\_Earithmic\_E\_2B \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \quad (15)$$

**Definition 15** We define  $c\_Earithmic\_EBIT1$  to be  $\lambda V0n \in ty\_Enum\_Enum.(ap\ (ap\ c\_Earithmic$

**Definition 16** We define  $c\_Earithmic\_ENUMERAL$  to be  $\lambda V0x \in ty\_Enum\_Enum.V0x$ .

**Definition 17** We define  $c\_Earithmic\_EBIT2$  to be  $\lambda V0n \in ty\_Enum\_Enum.(ap\ (ap\ c\_Earithmic$

Let  $c\_Earithmic\_EEXP : \iota$  be given. Assume the following.

$$c\_Earithmic\_EEXP \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \quad (16)$$

Let  $c\_Earithmic\_EDIV : \iota$  be given. Assume the following.

$$c\_Earithmic\_EDIV \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \quad (17)$$

**Definition 18** We define  $c\_Ebit\_EDIV\_EXP$  to be  $\lambda V0x \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$

Let  $c\_Earithmic\_E\_2D : \iota$  be given. Assume the following.

$$c\_Earithmic\_E\_2D \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \quad (18)$$

Let  $c\_Earithmic\_EMOD : \iota$  be given. Assume the following.

$$c\_Earithmic\_EMOD \in ((ty\_Enum\_Enum^{ty\_Enum\_Enum})^{ty\_Enum\_Enum}) \quad (19)$$

**Definition 19** We define  $c\_Ebit\_EMOD\_EXP$  to be  $\lambda V0x \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum$

**Definition 20** We define  $c\_Ebit\_EBITS$  to be  $\lambda V0h \in ty\_Enum\_Enum.\lambda V1l \in ty\_Enum\_Enum.\lambda V$

**Definition 21** We define  $c\_Ebit\_EBIT$  to be  $\lambda V0b \in ty\_Enum\_Enum.\lambda V1n \in ty\_Enum\_Enum.(ap$

Let  $ty\_Efc\_Efinite\_image : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_Efc\_Efinite\_image\ A0) \quad (20)$$

Let  $c\_Efc\_Edimindex : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_Efc\_Edimindex\ A\_27a \in (ty\_Enum\_Enum^{(ty\_Ebool\_Eitself\ A\_27a)}) \quad (21)$$

**Definition 22** We define  $c\_Ebool\_E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_Ebool\_E\_21\ 2)\ (\lambda V2t \in$

**Definition 23** We define  $c\_Ebool\_E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_Emin\_E\_40$

**Definition 24** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum$

**Definition 25** We define  $c\_2Ebool\_2E\_3F\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap c\_2Ebool\_2E\_2F\_5C$

**Definition 26** We define  $c\_2Efcp\_2Efinite\_index$  to be  $\lambda A\_27a : \iota.(ap (c\_2Emin\_2E\_40 (A\_27a^{ty\_2Enum\_2Enum}$

Let  $ty\_2Efcp\_2Ecart : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Efcp\_2Ecart A0 A1) \quad (22)$$

Let  $c\_2Efcp\_2Edest\_cart : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Efcp\_2Edest\_cart A\_27a A\_27b \in ((A\_27a^{(ty\_2Efcp\_2Efinite\_image A\_27b)})^{(ty\_2Efcp\_2Ecart A\_27a A\_27b)}) \quad (23)$$

**Definition 27** We define  $c\_2Efcp\_2Efcp\_index$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in (ty\_2Efcp\_2Ecart A\_27a$

**Definition 28** We define  $c\_2Efcp\_2EFCP$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0g \in (A\_27a^{ty\_2Enum\_2Enum}).(ap$

**Definition 29** We define  $c\_2Ewords\_2En2w$  to be  $\lambda A\_27a : \iota.\lambda V0n \in ty\_2Enum\_2Enum.(ap (c\_2Efcp\_2EFC$

Let  $c\_2Einteger\_2Etint\_neg : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_neg \in ((ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)}) \quad (24)$$

Let  $c\_2Einteger\_2Etint\_eq : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Etint\_eq \in ((2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum)}) \quad (25)$$

Let  $c\_2Einteger\_2Eint\_ABS\_CLASS : \iota$  be given. Assume the following.

$$c\_2Einteger\_2Eint\_ABS\_CLASS \in (ty\_2Einteger\_2Eint)^{(2^{(ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)})} \quad (26)$$

**Definition 30** We define  $c\_2Einteger\_2Eint\_ABS$  to be  $\lambda V0r \in (ty\_2Epair\_2Eprod ty\_2Enum\_2Enum ty\_2Enum\_2Enum)$

**Definition 31** We define  $c\_2Einteger\_2Eint\_neg$  to be  $\lambda V0T1 \in ty\_2Einteger\_2Eint.(ap c\_2Einteger\_2Eint$

**Definition 32** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

**Definition 33** We define  $c\_2Ebit\_2ESBIT$  to be  $\lambda V0b \in 2.\lambda V1n \in ty\_2Enum\_2Enum.(ap (ap (ap (c\_2Ebo$

Let  $c\_2Esum\_num\_2ESUM : \iota$  be given. Assume the following.

$$c\_2Esum\_num\_2ESUM \in ((ty\_2Enum\_2Enum)^{(ty\_2Enum\_2Enum)^{(ty\_2Enum\_2Enum)^{(ty\_2Enum\_2Enum)}}})^{ty\_2Enum\_2Enum} \quad (27)$$

**Definition 34** We define  $c\_2Ewords\_2Ew2n$  to be  $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Efcp\_2Ecart 2 A\_27a).(ap (ap$



Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (34)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow \neg(p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow \neg( \\ & p\ V0t)))))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p\ V0t1) \Rightarrow \\ & ((p\ V1t2) \Rightarrow (p\ V2t3))) \Leftrightarrow (((p\ V0t1) \wedge (p\ V1t2)) \Rightarrow (p\ V2t3)))))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0i \in ty\_2Einteger\_2Eint. \\ & (((p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le\ (ap\ (c\_2Einteger\_word\_2EINT\_MIN \\ & A\_27a)\ (c\_2Ebool\_2Ethe\_value\ A\_27a)))\ V0i)) \wedge (p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le \\ & V0i)\ (ap\ (c\_2Einteger\_word\_2EINT\_MAX\ A\_27a)\ (c\_2Ebool\_2Ethe\_value \\ & A\_27a)))))) \Rightarrow ((ap\ (c\_2Einteger\_word\_2Ew2i\ A\_27a)\ (ap\ (c\_2Einteger\_word\_2Ei2w \\ & A\_27a)\ V0i)) = V0i))) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0w \in (ty\_2Efc\_2Ecart \\ & 2\ A\_27a). (\exists V1i \in ty\_2Einteger\_2Eint. ((V0w = (ap\ (c\_2Einteger\_word\_2Ei2w \\ & A\_27a)\ V1i)) \wedge ((p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le\ (ap\ (c\_2Einteger\_word\_2EINT\_MIN \\ & A\_27a)\ (c\_2Ebool\_2Ethe\_value\ A\_27a)))\ V1i)) \wedge (p\ (ap\ (ap\ c\_2Einteger\_2Eint\_le \\ & V1i)\ (ap\ (c\_2Einteger\_word\_2EINT\_MAX\ A\_27a)\ (c\_2Ebool\_2Ethe\_value \\ & A\_27a))))))))) \end{aligned} \quad (38)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0a \in (ty\_2Efc\_2Ecart \\ & 2\ A\_27a). (\forall V1b \in (ty\_2Efc\_2Ecart\ 2\ A\_27a). ((p\ (ap\ (ap \\ & (c\_2Ewords\_2Eword\_lt\ A\_27a)\ V0a)\ V1b)) \Leftrightarrow (p\ (ap\ (ap\ c\_2Einteger\_2Eint\_lt \\ & (ap\ (c\_2Einteger\_word\_2Ew2i\ A\_27a)\ V0a))\ (ap\ (c\_2Einteger\_word\_2Ew2i \\ & A\_27a)\ V1b)))))) \end{aligned} \quad (39)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow ((ap\ (c\_2Einteger\_word\_2Ew2i \\ & A\_27a)\ (ap\ (c\_2Ewords\_2Ew2w\ A\_27a)\ c\_2Enum\_2E0)) = (ap\ c\_2Einteger\_2Eint\_of\_num \\ & c\_2Enum\_2E0)) \end{aligned} \quad (40)$$

**Theorem 1**

$$\forall A_{27a}.nonempty\ A_{27a} \Rightarrow (\forall V0w \in (ty\_2EfcP\_2Ecart\ 2\ A_{27a}).((p\ (ap\ (ap\ c\_2Einteger\_2Eint\_lt\ (ap\ (c\_2Einteger\_word\_2Ew2i\ A_{27a})\ V0w))\ (ap\ c\_2Einteger\_2Eint\_of\_num\ c\_2Enum\_2E0)))) \Leftrightarrow (p\ (ap\ (ap\ (c\_2Ewords\_2Eword\_lt\ A_{27a})\ V0w)\ (ap\ (c\_2Ewords\_2En2w\ A_{27a})\ c\_2Enum\_2E0))))))$$