

thm_2Eintegral_2ERSUM__BOUND
(TMQvL32d5DGa7F1c4weourDYau7tA1FyVZE)

October 26, 2020

Definition 1 We define `c_2Emin_2E_3D` to be $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj_o } (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define `c_2Ebool_2E_2T` to be $(\text{ap } (\text{ap } (\text{c_2Emin_2E_3D } (2^2))) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x)$

Definition 3 We define `c_2Emin_2E_40` to be $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. \text{ap } P x)$ **then** (the $(\lambda x. x \in A \wedge P x)$ of type $\iota \Rightarrow \iota$).

Definition 4 We define `c_2Ebool_2E_3F` to be $\lambda A. \lambda P \in (2^{A \rightarrow 2}). (\text{ap } P (\text{ap } (\text{c_2Emin_2E_40 } A)))$

Definition 5 We define `c_2Ecombin_2E_2EK` to be $\lambda A. \lambda P \in (2^{A \rightarrow 2}). (\lambda V0x \in A. \lambda V1y \in A. P (V0x, V1y))$

Definition 6 We define `c_2Ecombin_2E_2ES` to be $\lambda A. \lambda P \in (2^{A \rightarrow 2}). (\lambda V0f \in ((A \rightarrow 2)^{A \rightarrow 2}). (\lambda V1x \in A. \lambda V2y \in A. P (V1x, V2y, f)))$

Definition 7 We define `c_2Ecombin_2E_2EI` to be $\lambda A. \lambda P \in (2^{A \rightarrow 2}). (\text{ap } (\text{ap } (\text{c_2Ecombin_2E_2ES } A) P))$

Let `ty_2Ehreal_2Ehreal` : ι be given. Assume the following.

$$\text{nonempty } \text{ty_2Ehreal_2Ehreal} \tag{1}$$

Let `ty_2Epair_2Eprod` : $\iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \forall A1. \text{nonempty } A1 \Rightarrow \text{nonempty } (\text{ty_2Epair_2Eprod } A0 A1) \tag{2}$$

Let `ty_2Erealax_2Ereal` : ι be given. Assume the following.

$$\text{nonempty } \text{ty_2Erealax_2Ereal} \tag{3}$$

Let `c_2Erealax_2Ereal__REP__CLASS` : ι be given. Assume the following.

$$\text{c_2Erealax_2Ereal_REP_CLASS} \in ((2^{(\text{ty_2Epair_2Eprod } \text{ty_2Ehreal_2Ehreal } \text{ty_2Ehreal_2Ehreal})} \text{ty_2Erealax_2Ereal})) \tag{4}$$

Definition 8 We define `c_2Ebool_2E_21` to be $\lambda A. \lambda P \in (2^{A \rightarrow 2}). (\text{ap } (\text{ap } (\text{c_2Emin_2E_3D } (2^{A \rightarrow 2}))) P)$

Definition 9 We define $c_2Erealax_2Ereal_REP$ to be $\lambda V0a \in ty_2Erealax_2Ereal.(ap (c_2Emin_2E.40 (ty$

Let $c_2Erealax_2Etreal_neg : \iota$ be given. Assume the following.

$$c_2Erealax_2Etreal_neg \in ((ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty_2Ehreal_2Ehreal)(ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty_2Ehreal_2Ehreal)) \quad (5)$$

Let $c_2Erealax_2Etreal_eq : \iota$ be given. Assume the following.

$$c_2Erealax_2Etreal_eq \in ((2^{(ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty_2Ehreal_2Ehreal)})(ty_2Epair_2Eprod ty_2Ehreal_2Ehreal)) \quad (6)$$

Let $c_2Erealax_2Ereal_ABS_CLASS : \iota$ be given. Assume the following.

$$c_2Erealax_2Ereal_ABS_CLASS \in (ty_2Erealax_2Ereal)^{(2^{(ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty_2Ehreal_2Ehreal)})} \quad (7)$$

Definition 10 We define $c_2Erealax_2Ereal_ABS$ to be $\lambda V0r \in (ty_2Epair_2Eprod ty_2Ehreal_2Ehreal ty$

Definition 11 We define $c_2Erealax_2Ereal_neg$ to be $\lambda V0T1 \in ty_2Erealax_2Ereal.(ap c_2Erealax_2Ereal$

Definition 12 We define c_2Ebool_2EF to be $(ap (c_2Ebool_2E.21 2) (\lambda V0t \in 2.V0t))$.

Definition 13 We define $c_2Emin_2E.3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 14 We define $c_2Ebool_2E.2F.5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E.21 2) (\lambda V2t \in$

Definition 15 We define c_2Ebool_2ECOND to be $\lambda A.27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A.27a.(\lambda V2t2 \in A.27a.($

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty ty_2Enum_2Enum \quad (8)$$

Let $c_2Earithmetic_2E.2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E.2B \in ((ty_2Enum_2Enum)^{ty_2Enum_2Enum})^{ty_2Enum_2Enum} \quad (9)$$

Definition 16 We define $c_2Ebool_2E.7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E.3D_3D_3E V0t) c_2Ebool_2E$

Let $c_2Enum_2EREP_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (omega^{ty_2Enum_2Enum}) \quad (10)$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (omega^{omega}) \quad (11)$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{omega}) \quad (12)$$

Definition 17 We define c_Enum_ESUC to be $\lambda V0m \in ty_Enum_Enum.(ap\ c_Enum_EABS_num$

Definition 18 We define $c_Eprim_rec_E_3C$ to be $\lambda V0m \in ty_Enum_Enum.\lambda V1n \in ty_Enum_Enum$

Definition 19 We define $c_Ebool_E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c_Ebool_E_21\ 2)\ (\lambda V2t \in$

Definition 20 We define $c_Earithmic_E_3C_3D$ to be $\lambda V0m \in ty_Enum_Enum.\lambda V1n \in ty_Enum_Enum$

Let $c_Enum_EZERO_REP : \iota$ be given. Assume the following.

$$c_Enum_EZERO_REP \in \omega \tag{13}$$

Definition 21 We define c_Enum_E0 to be $(ap\ c_Enum_EABS_num\ c_Enum_EZERO_REP)$.

Let $c_Ereal_Ereal_of_num : \iota$ be given. Assume the following.

$$c_Ereal_Ereal_of_num \in (ty_Erealax_Ereal^{ty_Enum_Enum}) \tag{14}$$

Let $c_Erealax_Etrealm : \iota$ be given. Assume the following.

$$c_Erealax_Etrealm \in ((2^{(ty_Epair_Eprod\ ty_Ehreal_Ehreal\ ty_Ehreal_Ehreal)})(ty_Epair_Eprod\ ty_Ehreal_Ehreal)) \tag{15}$$

Definition 22 We define $c_Erealax_Ereal_lt$ to be $\lambda V0T1 \in ty_Erealax_Ereal.\lambda V1T2 \in ty_Erealax_Ereal$

Definition 23 We define $c_Ereal_Ereal_lte$ to be $\lambda V0x \in ty_Erealax_Ereal.\lambda V1y \in ty_Erealax_Ereal$

Definition 24 We define c_Ereal_Eabs to be $\lambda V0x \in ty_Erealax_Ereal.(ap\ (ap\ (ap\ (c_Ebool_ECON$

Let $c_Etransc_Etdiv : \iota$ be given. Assume the following.

$$c_Etransc_Etdiv \in ((2^{(ty_Epair_Eprod\ (ty_Erealax_Ereal^{ty_Enum_Enum})\ (ty_Erealax_Ereal^{ty_Enum_Enum}))}) \tag{16}$$

Let $c_Erealax_Etrealm_add : \iota$ be given. Assume the following.

$$c_Erealax_Etrealm_add \in (((ty_Epair_Eprod\ ty_Ehreal_Ehreal\ ty_Ehreal_Ehreal)^{(ty_Epair_Eprod\ ty_Ehreal_Ehreal\ ty_Ehreal_Ehreal)})(ty_Epair_Eprod\ ty_Ehreal_Ehreal)) \tag{17}$$

Definition 25 We define $c_Erealax_Ereal_add$ to be $\lambda V0T1 \in ty_Erealax_Ereal.\lambda V1T2 \in ty_Erealax_Ereal$

Definition 26 We define $c_Ereal_Ereal_sub$ to be $\lambda V0x \in ty_Erealax_Ereal.\lambda V1y \in ty_Erealax_Ereal$

Let $c_Erealax_Etrealm_mul : \iota$ be given. Assume the following.

$$c_Erealax_Etrealm_mul \in (((ty_Epair_Eprod\ ty_Ehreal_Ehreal\ ty_Ehreal_Ehreal)^{(ty_Epair_Eprod\ ty_Ehreal_Ehreal\ ty_Ehreal_Ehreal)})(ty_Epair_Eprod\ ty_Ehreal_Ehreal)) \tag{18}$$

Definition 27 We define $c_Erealax_Ereal_mul$ to be $\lambda V0T1 \in ty_Erealax_Ereal.\lambda V1T2 \in ty_Erealax_Ereal$

Let $c_2Ereal_2Esum : \iota$ be given. Assume the following.

$$c_2Ereal_2Esum \in ((ty_2Erealax_2Ereal^{(ty_2Erealax_2Ereal^{ty_2Eenum_2Eenum})})^{(ty_2Epair_2Eprod\ ty_2Eenum_2Eenum)}) \quad (19)$$

Let $c_2Etransc_2Ersum : \iota$ be given. Assume the following.

$$c_2Etransc_2Ersum \in ((ty_2Erealax_2Ereal^{(ty_2Erealax_2Ereal^{ty_2Erealax_2Ereal})})^{(ty_2Epair_2Eprod\ (ty_2Erealax_2Ereal^{ty_2Eenum_2Eenum})})}) \quad (20)$$

Definition 28 We define $c_2Earithmetic_2E_3E$ to be $\lambda V0m \in ty_2Eenum_2Eenum.\lambda V1n \in ty_2Eenum_2Eenum.$

Definition 29 We define $c_2Earithmetic_2E_3E_3D$ to be $\lambda V0m \in ty_2Eenum_2Eenum.\lambda V1n \in ty_2Eenum_2Eenum.$

Definition 30 We define $c_2Etransc_2Edsize$ to be $\lambda V0D \in (ty_2Erealax_2Ereal^{ty_2Eenum_2Eenum}).(ap\ (c_2Erealax_2Ereal^{(2^{A_27b})^{A_27a}}))$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod \\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \end{aligned} \quad (21)$$

Definition 31 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap\ (c_2Erealax_2Ereal^{(2^{A_27b})^{A_27a}}))$

Let $c_2Etransc_2Edivision : \iota$ be given. Assume the following.

$$c_2Etransc_2Edivision \in ((2^{(ty_2Erealax_2Ereal^{ty_2Eenum_2Eenum})})^{(ty_2Epair_2Eprod\ ty_2Erealax_2Ereal\ ty_2Erealax_2Ereal\ ty_2Eenum_2Eenum)}) \quad (22)$$

Assume the following.

$$\begin{aligned} (\forall V0m \in ty_2Eenum_2Eenum.(\forall V1n \in ty_2Eenum_2Eenum.(\\ ((ap\ (ap\ c_2Earithmetic_2E_2B\ c_2Enum_2E0)\ V0m) = V0m) \wedge (((ap\ (\\ ap\ c_2Earithmetic_2E_2B\ V0m)\ c_2Enum_2E0) = V0m) \wedge (((ap\ (ap\ c_2Earithmetic_2E_2B \\ (ap\ c_2Enum_2ESUC\ V0m))\ V1n) = (ap\ c_2Enum_2ESUC\ (ap\ (ap\ c_2Earithmetic_2E_2B \\ V0m)\ V1n))) \wedge ((ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ (ap\ c_2Enum_2ESUC \\ V1n)) = (ap\ c_2Enum_2ESUC\ (ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ V1n)))))))))) \end{aligned} \quad (23)$$

Assume the following.

$$True \quad (24)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (25)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \quad (26)$$

Assume the following.

$$(\forall V0t \in 2.((p\ V0t) \vee \neg(p\ V0t))) \quad (27)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A_27a.(p\ V0t)) \Leftrightarrow (p\ V0t))) \quad (28)$$

Assume the following.

$$(\forall V0t \in 2. ((p\ V0t) \Rightarrow False) \Rightarrow (\neg(p\ V0t))) \quad (29)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(p\ V0t)) \Rightarrow ((p\ V0t) \Rightarrow False))) \quad (30)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge \\ & (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \quad (31) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \vee (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \vee True) \Leftrightarrow True) \wedge \\ & (((False \vee (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee False) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \vee \\ & (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \quad (32) \end{aligned}$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \Rightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge (((p\ V0t) \Rightarrow (p\ V0t)) \Leftrightarrow True) \wedge ((\\ & (p\ V0t) \Rightarrow False) \Leftrightarrow (\neg(p\ V0t)))))) \quad (33) \end{aligned}$$

Assume the following.

$$\begin{aligned} & ((\forall V0t \in 2. ((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge \\ & ((\neg False) \Leftrightarrow True))) \quad (34) \end{aligned}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (35)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (36)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2. (((True \Leftrightarrow (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \Leftrightarrow True) \Leftrightarrow \\ & (p\ V0t)) \wedge (((False \Leftrightarrow (p\ V0t)) \Leftrightarrow (\neg(p\ V0t))) \wedge (((p\ V0t) \Leftrightarrow False) \Leftrightarrow (\neg \\ & p\ V0t)))))) \quad (37) \end{aligned}$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty A.27a \Rightarrow (\forall V0t1 \in A.27a. (\forall V1t2 \in \\ A.27a. ((\text{ap } (\text{ap } (\text{ap } (\text{c.2Ebool.2ECOND } A.27a) \text{ c.2Ebool.2ET}) V0t1) \\ V1t2) = V0t1) \wedge ((\text{ap } (\text{ap } (\text{ap } (\text{c.2Ebool.2ECOND } A.27a) \text{ c.2Ebool.2EF}) \\ V0t1) V1t2) = V1t2)))))) \end{aligned} \quad (38)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). ((\neg (\forall V1x \in \\ A.27a. (p (\text{ap } V0P V1x)))))) \Leftrightarrow (\exists V2x \in A.27a. (\neg (p (\text{ap } V0P V2x)))))) \quad (39)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). (\forall V1Q \in \\ (2^{A.27a}). ((\forall V2x \in A.27a. ((p (\text{ap } V0P V2x)) \wedge (p (\text{ap } V1Q V2x)))))) \Leftrightarrow \\ ((\forall V3x \in A.27a. (p (\text{ap } V0P V3x))) \wedge (\forall V4x \in A.27a. (p (\\ \text{ap } V1Q V4x)))))) \end{aligned} \quad (40)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2. (\forall V1Q \in (\\ 2^{A.27a}). ((\text{ap } V0P) \wedge (\forall V2x \in A.27a. (p (\text{ap } V1Q V2x)))))) \Leftrightarrow (\forall V3x \in \\ A.27a. ((\text{ap } V0P) \wedge (p (\text{ap } V1Q V3x)))))) \end{aligned} \quad (41)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). (\forall V1Q \in \\ (2^{A.27a}). ((\exists V2x \in A.27a. ((p (\text{ap } V0P V2x)) \vee (p (\text{ap } V1Q V2x)))))) \Leftrightarrow \\ ((\exists V3x \in A.27a. (p (\text{ap } V0P V3x))) \vee (\exists V4x \in A.27a. (p (\\ \text{ap } V1Q V4x)))))) \end{aligned} \quad (42)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2. (\forall V1Q \in (\\ 2^{A.27a}). ((\text{ap } V0P) \vee (\exists V2x \in A.27a. (p (\text{ap } V1Q V2x)))))) \Leftrightarrow (\exists V3x \in \\ A.27a. ((\text{ap } V0P) \vee (p (\text{ap } V1Q V3x)))))) \end{aligned} \quad (43)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). (\forall V1Q \in \\ 2. ((\exists V2x \in A.27a. ((p (\text{ap } V0P V2x)) \wedge (p V1Q)))))) \Leftrightarrow ((\exists V3x \in \\ A.27a. (p (\text{ap } V0P V3x)) \wedge (p V1Q)))) \end{aligned} \quad (44)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2. (\forall V1Q \in (\\ 2^{A.27a}). ((\forall V2x \in A.27a. ((\text{ap } V0P) \vee (p (\text{ap } V1Q V2x)))))) \Leftrightarrow ((\text{ap } \\ V0P) \vee (\forall V3x \in A.27a. (p (\text{ap } V1Q V3x)))))) \end{aligned} \quad (45)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(\forall V2C \in 2.(((p V0A) \vee (p V1B) \vee (p V2C))) \Leftrightarrow (((p V0A) \vee (p V1B)) \vee (p V2C)))))) \quad (46)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((p V0A) \vee (p V1B)) \Leftrightarrow ((p V1B) \vee (p V0A)))))) \quad (47)$$

Assume the following.

$$(\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg(p V0A) \vee \neg(p V1B)))))) \wedge (((\neg((p V0A) \vee (p V1B))) \Leftrightarrow ((\neg(p V0A) \wedge \neg(p V1B)))))) \quad (48)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow \forall A.27b.nonempty \ A.27b \Rightarrow (\forall V0P \in ((2^{A.27b})^{A.27a}).((\forall V1x \in A.27a.(\exists V2y \in A.27b.(p (ap (ap V0P V1x) V2y)))) \Leftrightarrow (\exists V3f \in (A.27b^{A.27a}).(\forall V4x \in A.27a.(p (ap (ap V0P V4x) (ap V3f V4x))))))) \quad (49)$$

Assume the following.

$$\forall A.27a.nonempty \ A.27a \Rightarrow (\forall V0x \in A.27a.((ap (c.2Ecombin.2EI A.27a) V0x) = V0x)) \quad (50)$$

Assume the following.

$$(\forall V0x \in ty.2Erealax.2Ereal.(\forall V1y \in ty.2Erealax.2Ereal.(\forall V2z \in ty.2Erealax.2Ereal.(((p (ap (ap c.2Ereal.2Ereal_lte V0x) V1y)) \wedge (p (ap (ap c.2Ereal.2Ereal_lte (ap c.2Ereal.2Ereal_of_num c.2Enum.2E0) V2z))) \Rightarrow (p (ap (ap c.2Ereal.2Ereal_lte (ap (ap c.2Erealax.2Ereal_mul V0x) V2z)) (ap (ap c.2Erealax.2Ereal_mul V1y) V2z)))))) \quad (51)$$

Assume the following.

$$(\forall V0x \in ty.2Erealax.2Ereal.(\forall V1y \in ty.2Erealax.2Ereal.(\forall V2z \in ty.2Erealax.2Ereal.(((p (ap (ap c.2Ereal.2Ereal_lte (ap c.2Ereal.2Ereal_of_num c.2Enum.2E0) V0x)) \wedge (p (ap (ap c.2Ereal.2Ereal_lte V1y) V2z))) \Rightarrow (p (ap (ap c.2Ereal.2Ereal_lte (ap (ap c.2Erealax.2Ereal_mul V0x) V1y)) (ap (ap c.2Erealax.2Ereal_mul V0x) V2z)))))) \quad (52)$$

Assume the following.

$$(\forall V0d \in (ty.2Erealax.2Ereal^{ty.2Enum.2Enum}).(\forall V1a \in ty.2Erealax.2Ereal.(\forall V2b \in ty.2Erealax.2Ereal.(((p (ap (ap c.2Etransc.2Edivision (ap (ap (c.2Epair.2E.2C ty.2Erealax.2Ereal ty.2Erealax.2Ereal) V1a) V2b)) V0d)) \Rightarrow (\forall V3n \in ty.2Enum.2Enum.(p (ap (ap c.2Ereal.2Ereal_lte (ap V0d V3n)) (ap V0d (ap c.2Enum.2ESUC V3n)))))))))) \quad (53)$$

Assume the following.

$$\begin{aligned}
& (\forall V0d \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V1m \in \\
& ty_2Enum_2Enum.(\forall V2n \in ty_2Enum_2Enum.((ap (ap c_2Ereal_2Esum \\
& (ap (ap (c_2Epair_2E_2C ty_2Enum_2Enum ty_2Enum_2Enum) V1m) V2n)) \\
& (\lambda V3i \in ty_2Enum_2Enum.(ap (ap c_2Ereal_2Ereal_sub (ap V0d \\
& (ap c_2Enum_2ESUC V3i))) (ap V0d V3i)))))) = (ap (ap c_2Ereal_2Ereal_sub \\
& (ap V0d (ap (ap c_2Earithmetic_2E_2B V1m) V2n)) (ap V0d V1m))))))
\end{aligned} \tag{54}$$

Assume the following.

$$(\forall V0x \in ty_2Erealax_2Ereal.(p (ap (ap c_2Ereal_2Ereal_lte V0x) V0x))) \tag{55}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Erealax_2Ereal.(\forall V1y \in ty_2Erealax_2Ereal. \\
& (\forall V2z \in ty_2Erealax_2Ereal.(((p (ap (ap c_2Ereal_2Ereal_lte \\
& V0x) V1y)) \wedge (p (ap (ap c_2Ereal_2Ereal_lte V1y) V2z)))) \Rightarrow (p (ap (\\
& ap c_2Ereal_2Ereal_lte V0x) V2z))))))
\end{aligned} \tag{56}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Erealax_2Ereal.(\forall V1y \in ty_2Erealax_2Ereal. \\
& ((p (ap (ap c_2Ereal_2Ereal_lte (ap c_2Ereal_2Ereal_of_num \\
& c_2Enum_2E0)) (ap (ap c_2Ereal_2Ereal_sub V0x) V1y))) \Leftrightarrow (p (ap \\
& (ap c_2Ereal_2Ereal_lte V1y) V0x))))))
\end{aligned} \tag{57}$$

Assume the following.

$$\begin{aligned}
& (\forall V0f \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V1g \in \\
& (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V2m \in ty_2Enum_2Enum. \\
& (\forall V3n \in ty_2Enum_2Enum.((\forall V4r \in ty_2Enum_2Enum. \\
& (((p (ap (ap c_2Earithmetic_2E_3C_3D V2m) V4r)) \wedge (p (ap (ap c_2Eprim_rec_2E_3C \\
& V4r) (ap (ap c_2Earithmetic_2E_2B V3n) V2m)))) \Rightarrow (p (ap (ap c_2Ereal_2Ereal_lte \\
& (ap V0f V4r) (ap V1g V4r)))))) \Rightarrow (p (ap (ap c_2Ereal_2Ereal_lte (\\
& ap (ap c_2Ereal_2Esum (ap (ap (c_2Epair_2E_2C ty_2Enum_2Enum ty_2Enum_2Enum) \\
& V2m) V3n)) V0f)) (ap (ap c_2Ereal_2Esum (ap (ap (c_2Epair_2E_2C \\
& ty_2Enum_2Enum ty_2Enum_2Enum) V2m) V3n)) V1g)))))))))
\end{aligned} \tag{58}$$

Assume the following.

$$\begin{aligned}
& (\forall V0f \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V1m \in \\
& ty_2Enum_2Enum.(\forall V2n \in ty_2Enum_2Enum.(p (ap (ap c_2Ereal_2Ereal_lte \\
& (ap c_2Ereal_2Eabs (ap (ap c_2Ereal_2Esum (ap (ap (c_2Epair_2E_2C \\
& ty_2Enum_2Enum ty_2Enum_2Enum) V1m) V2n)) V0f))) (ap (ap c_2Ereal_2Esum \\
& (ap (ap (c_2Epair_2E_2C ty_2Enum_2Enum ty_2Enum_2Enum) V1m) V2n)) \\
& (\lambda V3n \in ty_2Enum_2Enum.(ap c_2Ereal_2Eabs (ap V0f V3n))))))))))
\end{aligned} \tag{59}$$

Assume the following.

$$\begin{aligned}
& (\forall V0f \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V1c \in \\
& \quad ty_2Erealax_2Ereal.(\forall V2m \in ty_2Enum_2Enum.(\forall V3n \in \\
& \quad \quad ty_2Enum_2Enum.((ap (ap c_2Ereal_2Esum (ap (ap (c_2Epair_2E_2C \\
& \quad \quad \quad ty_2Enum_2Enum ty_2Enum_2Enum) V2m) V3n)) (\lambda V4n \in ty_2Enum_2Enum. \\
& \quad \quad \quad (ap (ap c_2Erealax_2Ereal_mul V1c) (ap V0f V4n)))))) = (ap (ap c_2Erealax_2Ereal_mul \\
& \quad \quad \quad V1c) (ap (ap c_2Ereal_2Esum (ap (ap (c_2Epair_2E_2C ty_2Enum_2Enum \\
& \quad \quad \quad \quad ty_2Enum_2Enum) V2m) V3n)) V0f))))))
\end{aligned} \tag{60}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Erealax_2Ereal.(\forall V1y \in ty_2Erealax_2Ereal. \\
& \quad ((ap c_2Ereal_2Eabs (ap (ap c_2Erealax_2Ereal_mul V0x) V1y)) = \\
& \quad (ap (ap c_2Erealax_2Ereal_mul (ap c_2Ereal_2Eabs V0x)) (ap c_2Ereal_2Eabs \\
& \quad \quad \quad V1y))))))
\end{aligned} \tag{61}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in ty_2Erealax_2Ereal.(p (ap (ap c_2Ereal_2Ereal_lte \\
& \quad (ap c_2Ereal_2Ereal_of_num c_2Enum_2E0)) (ap c_2Ereal_2Eabs \\
& \quad \quad \quad V0x))))))
\end{aligned} \tag{62}$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \tag{63}$$

Assume the following.

$$(\forall V0A \in 2.((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \tag{64}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& \quad ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False))))))
\end{aligned} \tag{65}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& \quad ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False))))))
\end{aligned} \tag{66}$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \tag{67}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow (\\
& \quad (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg \\
& \quad p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\
& \quad \quad \quad ((\neg(p V1q)) \vee (\neg(p V0p))))))))))
\end{aligned} \tag{68}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \vee V0p) \Leftrightarrow (\\
& (p \vee V1q) \wedge (p \vee V2r))) \Leftrightarrow (((p \vee V0p) \vee (\neg(p \vee V1q)) \vee \neg(p \vee V2r)))) \wedge (((p \vee V1q) \vee \\
& (\neg(p \vee V0p))) \wedge ((p \vee V2r) \vee \neg(p \vee V0p))))))
\end{aligned} \tag{69}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \vee V0p) \Leftrightarrow (\\
& (p \vee V1q) \vee (p \vee V2r))) \Leftrightarrow (((p \vee V0p) \vee \neg(p \vee V1q)) \wedge ((p \vee V0p) \vee \neg(p \vee V2r))) \wedge \\
& ((p \vee V1q) \vee ((p \vee V2r) \vee \neg(p \vee V0p))))))
\end{aligned} \tag{70}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p \vee V0p) \Leftrightarrow (\\
& (p \vee V1q) \Rightarrow (p \vee V2r))) \Leftrightarrow (((p \vee V0p) \vee (p \vee V1q)) \wedge (((p \vee V0p) \vee \neg(p \vee V2r))) \wedge (\\
& \neg(p \vee V1q) \vee ((p \vee V2r) \vee \neg(p \vee V0p))))))
\end{aligned} \tag{71}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (((p \vee V0p) \Leftrightarrow \neg(p \vee V1q)) \Leftrightarrow (((p \vee V0p) \vee \\
& (p \vee V1q)) \wedge (\neg(p \vee V1q) \vee \neg(p \vee V0p))))))
\end{aligned} \tag{72}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \vee V0p) \Rightarrow (p \vee V1q))) \Rightarrow (p \vee V0p))) \tag{73}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \vee V0p) \Rightarrow (p \vee V1q))) \Rightarrow \neg(p \vee V1q))) \tag{74}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \vee V0p) \vee (p \vee V1q))) \Rightarrow \neg(p \vee V0p))) \tag{75}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. ((\neg((p \vee V0p) \vee (p \vee V1q))) \Rightarrow \neg(p \vee V1q))) \tag{76}$$

Assume the following.

$$(\forall V0p \in 2. ((\neg(\neg(p \vee V0p))) \Rightarrow (p \vee V0p))) \tag{77}$$

Assume the following.

$$\begin{aligned}
& (\forall V0a \in ty_2Erealax_2Ereal. (\forall V1b \in ty_2Erealax_2Ereal. \\
& \quad (\forall V2D \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). (\forall V3p \in \\
& \quad (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). ((p (ap (ap c_2Etransc_2Etdiv \\
& \quad (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal ty_2Erealax_2Ereal) \\
V0a) V1b)) (ap (ap (c_2Epair_2E_2C (ty_2Erealax_2Ereal^{ty_2Enum_2Enum} \\
& \quad (ty_2Erealax_2Ereal^{ty_2Enum_2Enum})) V2D) V3p))) \Leftrightarrow ((p (ap (ap \\
& \quad c_2Etransc_2Edivision (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal \\
& \quad ty_2Erealax_2Ereal) V0a) V1b)) V2D)) \wedge (\forall V4n \in ty_2Enum_2Enum. \\
& \quad ((p (ap (ap c_2Ereal_2Ereal_lte (ap V2D V4n)) (ap V3p V4n))) \wedge (p \\
& \quad (ap (ap c_2Ereal_2Ereal_lte (ap V3p V4n)) (ap V2D (ap c_2Enum_2ESUC \\
& \quad V4n))))))))))
\end{aligned} \tag{78}$$

Assume the following.

$$\begin{aligned}
& (\forall V0D \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). (\forall V1p \in \\
& \quad (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). (\forall V2f \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). \\
& \quad ((ap (ap c_2Etransc_2Ersum (ap (ap (c_2Epair_2E_2C (ty_2Erealax_2Ereal^{ty_2Enum_2Enum} \\
& \quad (ty_2Erealax_2Ereal^{ty_2Enum_2Enum})) V0D) V1p)) V2f) = (ap (ap \\
& \quad c_2Ereal_2Esum (ap (ap (c_2Epair_2E_2C ty_2Enum_2Enum ty_2Enum_2Enum) \\
& \quad c_2Enum_2E0) (ap c_2Etransc_2Edsize V0D))) (\lambda V3n \in ty_2Enum_2Enum. \\
& \quad (ap (ap c_2Erealax_2Ereal_mul (ap V2f (ap V1p V3n))) (ap (ap c_2Ereal_2Ereal_sub \\
& \quad (ap V0D (ap c_2Enum_2ESUC V3n))) (ap V0D V3n))))))
\end{aligned} \tag{79}$$

Assume the following.

$$\begin{aligned}
& (\forall V0D \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). (\forall V1a \in \\
& \quad ty_2Erealax_2Ereal. (\forall V2b \in ty_2Erealax_2Ereal. ((p (ap \\
& \quad (ap c_2Etransc_2Edivision (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal \\
& \quad ty_2Erealax_2Ereal) V1a) V2b)) V0D)) \Rightarrow ((ap V0D c_2Enum_2E0) = V1a))))
\end{aligned} \tag{80}$$

Assume the following.

$$\begin{aligned}
& (\forall V0D \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). (\forall V1a \in \\
& \quad ty_2Erealax_2Ereal. (\forall V2b \in ty_2Erealax_2Ereal. ((p (ap \\
& \quad (ap c_2Etransc_2Edivision (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal \\
& \quad ty_2Erealax_2Ereal) V1a) V2b)) V0D)) \Rightarrow ((ap V0D (ap c_2Etransc_2Edsize \\
& \quad V0D)) = V2b))))
\end{aligned} \tag{81}$$

Assume the following.

$$\begin{aligned}
& (\forall V0D \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}). (\forall V1a \in \\
& \quad ty_2Erealax_2Ereal. (\forall V2b \in ty_2Erealax_2Ereal. ((p (ap \\
& \quad (ap c_2Etransc_2Edivision (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal \\
& \quad ty_2Erealax_2Ereal) V1a) V2b)) V0D)) \Rightarrow (p (ap (ap c_2Ereal_2Ereal_lte \\
& \quad V1a) V2b))))))
\end{aligned} \tag{82}$$

Assume the following.

$$\begin{aligned}
& (\forall V0D \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V1a \in \\
& ty_2Erealax_2Ereal.(\forall V2b \in ty_2Erealax_2Ereal.((p (ap \\
& (ap c_2Etransc_2Edivision (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal \\
& ty_2Erealax_2Ereal) V1a) V2b)) V0D)) \Rightarrow (\forall V3r \in ty_2Enum_2Enum. \\
& (p (ap (ap c_2Ereal_2Ereal_lte V1a) (ap V0D V3r)))))))))
\end{aligned} \tag{83}$$

Assume the following.

$$\begin{aligned}
& (\forall V0D \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V1a \in \\
& ty_2Erealax_2Ereal.(\forall V2b \in ty_2Erealax_2Ereal.((p (ap \\
& (ap c_2Etransc_2Edivision (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal \\
& ty_2Erealax_2Ereal) V1a) V2b)) V0D)) \Rightarrow (\forall V3r \in ty_2Enum_2Enum. \\
& (p (ap (ap c_2Ereal_2Ereal_lte (ap V0D V3r)) V2b)))))))))
\end{aligned} \tag{84}$$

Theorem 1

$$\begin{aligned}
& (\forall V0a \in ty_2Erealax_2Ereal.(\forall V1b \in ty_2Erealax_2Ereal. \\
& (\forall V2d \in (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V3p \in \\
& (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}).(\forall V4e \in ty_2Erealax_2Ereal. \\
& (\forall V5f \in (ty_2Erealax_2Ereal^{ty_2Erealax_2Ereal}).(((p \\
& (ap (ap c_2Etransc_2Etdiv (ap (ap (c_2Epair_2E_2C ty_2Erealax_2Ereal \\
& ty_2Erealax_2Ereal) V0a) V1b)) (ap (ap (c_2Epair_2E_2C (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}) \\
& (ty_2Erealax_2Ereal^{ty_2Enum_2Enum})) V2d) V3p))) \wedge (\forall V6x \in \\
& ty_2Erealax_2Ereal.(((p (ap (ap c_2Ereal_2Ereal_lte V0a) V6x)) \wedge \\
& (p (ap (ap c_2Ereal_2Ereal_lte V6x) V1b))) \Rightarrow (p (ap (ap c_2Ereal_2Ereal_lte \\
& (ap c_2Ereal_2Eabs (ap V5f V6x))) V4e)))))) \Rightarrow (p (ap (ap c_2Ereal_2Ereal_lte \\
& (ap c_2Ereal_2Eabs (ap (ap c_2Etransc_2Ersum (ap (ap (c_2Epair_2E_2C \\
& (ty_2Erealax_2Ereal^{ty_2Enum_2Enum}) (ty_2Erealax_2Ereal^{ty_2Enum_2Enum})) \\
& V2d) V3p)) V5f))) (ap (ap c_2Erealax_2Ereal_mul V4e) (ap (ap c_2Ereal_2Ereal_sub \\
& V1b) V0a)))))))))))))
\end{aligned}$$