

thm\_2Eiterate\_2ENSUM\_BOUND\_LT  
(TMF77AzBrj9hji9LVeN VYUkYRuV4agisGDj)

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Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ZERO\_REP \in \omega$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

*nonempty* *ty\_2Enum\_2Enum* (2)

Let  $c_2Enum_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{omega}) \quad (3)$$

**Definition 1** We define  $c_2Emin_2E_3D$  to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

**Definition 3** We define  $c\_2Ebool\_2ET$  to be  $(ap \ (ap \ (c\_2Emin\_2E\_3D \ (2^2)) \ (\lambda V0x \in 2.V0x)) \ (\lambda V1x \in 2.V1x))$

**Definition 4** We define  $c_{\leq 2E\text{bool}} \cdot 2E\cdot 21$  to be  $\lambda A \cdot 27a : \iota.(\lambda V0P \in (2^{A \cdot 27a}).(ap \ (ap \ (c_{\leq 2E\text{min}} \cdot 2E\cdot 3D \cdot (2^{A \cdot 27a})^{\iota} \cdot 2E\cdot 21) \ P) \ V) \ 0)$

**Definition 5** We define  $c\_2Ebool\_2EF$  to be  $(ap\ (c\_2Ebool\_2E_21\ 2)\ (\lambda V0t\in 2.V0t))$ .

**Definition 6** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p\ P \Rightarrow p\ Q)$  of type  $\iota$ .

**Definition 7** We define  $c_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap\ (ap\ c_2Emin\_3D\_3D\_3E\ V0t)\ c_2Ebool\_2EF))$

Let  $c_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (4)$$

Let  $c\text{-}2Enum\text{-}2ESUC\text{-}REP : \iota$  be given. Assume the following.

$$c_2Enym_2ESUC\_REP \in (\omega\omega\omega) \quad (5)$$

**Definition 9** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num\ (V0m))$

**Definition 10** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\text{if } (\exists x \in A.p\ (ap\ P\ x)) \text{ then } (\lambda x.x \in A \wedge P(x)) \text{ else } \perp$

**Definition 11** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40\ (A\_27a))\ (V0P))))$

**Definition 12** We define  $c\_2Eprim\_rec\_2E\_3C$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.(c\_2Eprim\_rec\ (V0m\ V1n))$

**Definition 13** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (V1t2))))$

**Definition 14** We define  $c\_2Earithmetic\_2E\_3C\_3D$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.\lambda V1n \in ty\_2Enum\_2Enum.(c\_2Earithmetic\_2E\_3C\_3D\ (V0m\ V1n))$

**Definition 15** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap\ V1f\ V0x)))$

Let  $c\_2Epred\_set\_2ECARD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow c\_2Epred\_set\_2ECARD\ A\_27a \in ((ty\_2Enum\_2Enum)^{(2^{A\_27a})}) \quad (6)$$

Let  $c\_2Earithmetic\_2E\_2A : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2A \in ((ty\_2Enum\_2Enum)^{(ty\_2Enum\_2Enum)}) \quad (7)$$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum)^{(ty\_2Enum\_2Enum)}) \quad (8)$$

**Definition 16** We define  $c\_2Eiterate\_2Eneutral$  to be  $\lambda A\_27a : \iota.\lambda V0op \in ((A\_27a)^{A\_27a})^{A\_27a}.$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.\text{nonempty } A0 \Rightarrow \forall A1.\text{nonempty } A1 \Rightarrow \text{nonempty } (ty\_2Epair\_2Eprod\ A0\ A1) \quad (9)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow \forall A\_27b.\text{nonempty } A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (10)$$

**Definition 17** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Epair\_2Eprod\ A\_27a\ A\_27b)\ (V0x\ V1y))$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow \forall A\_27b.\text{nonempty } A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{A\_27b})}) \quad (11)$$

**Definition 18** We define  $c\_2Eiterate\_2Esupport$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0op \in ((A\_27b)^{A\_27b})^{A\_27b}.\lambda V1op \in ((A\_27b)^{A\_27b})^{A\_27b}.$

**Definition 19** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

**Definition 20** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap (c\_2Epred\_set\_2EINSERT A\_27a) V0x V1s)$

**Definition 21** We define  $c\_2Epred\_set\_2EEMPTY$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2EF).$

**Definition 22** We define  $c\_2Epred\_set\_2EFINITE$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).(ap (c\_2Ebool\_2E\_21 (2^{A\_27a})) V0s)$

**Definition 23** We define  $c\_2Eiterate\_2EITSET$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in ((A\_27a^{A\_27a})^{A\_27b}).\lambda V0m (c\_2Eiterate\_2EITSET A\_27a V0f V0m)$

**Definition 24** We define  $c\_2Eiterate\_2Eiterate$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0op \in ((A\_27b^{A\_27b})^{A\_27b}).\lambda V0m (c\_2Eiterate\_2Eiterate A\_27a V0op V0m)$

**Definition 25** We define  $c\_2Eiterate\_2Enum$  to be  $\lambda A\_27a : \iota.(ap (c\_2Eiterate\_2Eiterate A\_27a ty\_2Enum\_2Enum) V0m)$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.(\forall V2p \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Eprim\_rec\_2E\_3C V0m) V1n)) \wedge (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V1n) V2p))) \Rightarrow (p (ap (ap c\_2Eprim\_rec\_2E\_3C V0m) V2p)))))) \quad (12)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum.(p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V0m) V0m))) \quad (13)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty\_2Enum\_2Enum.(\forall V1n \in ty\_2Enum\_2Enum.(\forall V2p \in ty\_2Enum\_2Enum.((p (ap (ap c\_2Earithmetic\_2E\_3C\_3D (ap (ap c\_2Earithmetic\_2E\_2A V0m) V1n)) (ap (ap c\_2Earithmetic\_2E\_2A V0m) V2p))) \Leftrightarrow ((V0m = c\_2Enum\_2E0) \vee (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D V1n) V2p))))))) \\ & \end{aligned} \quad (14)$$

Assume the following.

$$True \quad (15)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (16)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow ( \\ & \forall V0t1 \in A\_27a.(\forall V1t2 \in A\_27b.((ap (\lambda V2x \in A\_27b. \\ & V0t1) V1t2) = V0t1))) \end{aligned} \quad (17)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p V0t)) \Leftrightarrow (p V0t))) \quad (18)$$

Assume the following.

$$(\forall V0t \in 2.(((p V0t) \Rightarrow False) \Rightarrow (\neg(p V0t)))) \quad (19)$$

Assume the following.

$$(\forall V0t \in 2.((\neg(p V0t)) \Rightarrow ((p V0t) \Rightarrow False))) \quad (20)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow \\ & \quad (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge \\ & \quad (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (21)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \vee (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \vee True) \Leftrightarrow True) \wedge \\ & \quad (((False \vee (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee False) \Leftrightarrow (p V0t)) \wedge (((p V0t) \vee \\ & \quad (p V0t)) \Leftrightarrow (p V0t)))))) \end{aligned} \quad (22)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & \quad True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & \quad (p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t))))))) \end{aligned} \quad (23)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t)) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge \\ ((\neg False) \Leftrightarrow True)))) \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow \\ & \quad (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg \\ & \quad (p V0t))))))) \end{aligned} \quad (25)$$

Assume the following.

$$\forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0P \in (2^{A\_27a}).((\neg(\exists V1x \in \\ A\_27a.(p (ap V0P V1x)))) \Leftrightarrow (\forall V2x \in A\_27a.(\neg(p (ap V0P V2x))))))) \quad (26)$$

Assume the following.

$$\begin{aligned} & (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg( \\ p V0A)) \vee (\neg(p V1B)))) \wedge (((\neg((p V0A) \vee (p V1B))) \Leftrightarrow ((\neg(p V0A)) \wedge (\neg(p V1B))))))) \end{aligned} \quad (27)$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0f \in (ty\_2Enum\_2Enum^{A_{27a}}). \\
& \quad (\forall V1g \in (ty\_2Enum\_2Enum^{A_{27a}}). (\forall V2s \in (2^{A_{27a}}). \\
& \quad (((p (ap (c_2Epred_set_2EFINITE A_{27a}) V2s)) \wedge ((\forall V3x \in \\
& \quad A_{27a}. ((p (ap (ap (c_2Ebool_2EIN A_{27a}) V3x) V2s)) \Rightarrow (p (ap (ap c_2Earithmetic_2E_3C_3D \\
& \quad (ap V0f V3x) (ap V1g V3x))))))) \wedge (\exists V4x \in A_{27a}. ((p (ap (ap (c_2Ebool_2EIN \\
& \quad A_{27a}) V4x) V2s)) \wedge (p (ap (ap c_2Eprim_rec_2E_3C (ap V0f V4x) \\
& \quad ap V1g V4x))))))) \Rightarrow (p (ap (ap c_2Eprim_rec_2E_3C (ap (ap (c_2Eiterate_2Ensum \\
& \quad A_{27a}) V2s) V0f)) (ap (ap (c_2Eiterate_2Ensum A_{27a}) V2s) V1g))))))) \\
& \quad (28)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow (\forall V0c \in ty\_2Enum\_2Enum. \\
& \quad (\forall V1s \in (2^{A_{27a}}). ((p (ap (c_2Epred_set_2EFINITE A_{27a}) \\
& \quad V1s)) \Rightarrow ((ap (ap (c_2Eiterate_2Ensum A_{27a}) V1s) (\lambda V2n \in A_{27a}. \\
& \quad V0c)) = (ap (ap c_2Earithmetic_2E_2A (ap (c_2Epred_set_2ECARD \\
& \quad A_{27a}) V1s)) V0c))))))) \\
& \quad (29)
\end{aligned}$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (30)$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (31)$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& \quad ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \\
& \quad (32)
\end{aligned}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2. (\forall V1B \in 2. (((\neg((\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow \\
& \quad ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \\
& \quad (33)
\end{aligned}$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (34)$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow \\
& \quad (p V1q) \Leftrightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee ((\neg(p \\
& \quad V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee ((\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee \\
& \quad ((\neg(p V1q)) \vee (\neg(p V0p))))))))))) \\
& \quad (35)
\end{aligned}$$

Assume the following.

$$\begin{aligned} (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow \\ (p V1q) \wedge (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q)) \vee (\neg(p V2r)))) \wedge (((p V1q) \vee \\ (\neg(p V0p))) \wedge ((p V2r) \vee (\neg(p V0p))))))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned} (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow \\ (p V1q) \vee (p V2r))) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge \\ ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p V0p) \Leftrightarrow \\ (p V1q) \Rightarrow (p V2r))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge \\ (\neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \end{aligned} \quad (38)$$

Assume the following.

$$(\forall V0p \in 2.(\forall V1q \in 2.(((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee \\ (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p))))))) \quad (39)$$

### Theorem 1

$$\begin{aligned} \forall A\_27a.\text{nonempty } A\_27a \Rightarrow (\forall V0s \in (2^{A\_27a}).(\forall V1f \in \\ (ty\_2Enum\_2Enum^{A\_27a}).(\forall V2b \in ty\_2Enum\_2Enum.(((p (ap \\ (c\_2Epred\_set\_2EFINITE A\_27a) V0s)) \wedge ((\forall V3x \in A\_27a.\\ (p (ap (ap (c\_2Ebool\_2EIN A\_27a) V3x) V0s)) \Rightarrow (p (ap (ap c\_2Earithmetic\_2E\_3C\_3D \\ (ap V1f V3x)) V2b)))) \wedge (\exists V4x \in A\_27a.((p (ap (ap (c\_2Ebool\_2EIN \\ A\_27a) V4x) V0s)) \wedge (p (ap (ap c\_2Eprim\_rec\_2E\_3C (ap V1f V4x)) V2b))))))) \Rightarrow \\ (p (ap (ap c\_2Eprim\_rec\_2E\_3C (ap (ap (c\_2Eiterate\_2Ensum A\_27a) \\ V0s) V1f)) (ap (ap c\_2Earithmetic\_2E\_2A (ap (c\_2Epred\_set\_2ECARD \\ A\_27a) V0s)) V2b))))))) \end{aligned}$$