

# thm\_2Eordinal\_2Eadd\_\_omega\_\_islimit (TMXzQZpr6Rgo8gq2MbJDJjyAtjhGVNVrX9E)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2E\_21$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})))$

**Definition 4** We define  $c\_2Ebool\_2E\_21$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

Let  $ty\_2Eordinal\_2Eordinal : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty\_2Eordinal\_2Eordinal A0) \quad (1)$$

Let  $ty\_2Eenum\_2Eenum : \iota$  be given. Assume the following.

$$nonempty ty\_2Eenum\_2Eenum \quad (2)$$

Let  $c\_2Eordinal\_2EfromNat : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Eordinal\_2EfromNat A\_27a \in (ty\_2Eordinal\_2Eordinal A\_27a)^{ty\_2Eenum\_2Eenum} \quad (3)$$

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t)))$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \quad (4)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (5)$$

**Definition 7** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap (c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ 2)^{A\_27b}})$$
 (6)

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1)$$
 (7)

Let  $ty\_2Ewellorder\_2Ewellorder : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ewellorder\_2Ewellorder\ A0)$$
 (8)

Let  $c\_2Eordinal\_2Eordinal\_REP\_CLASS : \iota \Rightarrow \iota$  be given. Assume the following.

$$A\_27a \in ((2^{(ty\_2Ewellorder\_2Ewellorder\ (ty\_2Esum\_2Esum\ ty\_2Enum\_2Enum\ A\_27a))})^{(ty\_2Eordinal\_2Eordinal\ A\_27a)})$$
 (9)

**Definition 8** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A$ . **if**  $(\exists x \in A.p (ap\ P\ x))$  **then**  $(the\ (\lambda x.x \in A \wedge p\ of\ type\ \iota \Rightarrow \iota.$

**Definition 9** We define  $c\_2Eordinal\_2Eordinal\_REP$  to be  $\lambda A\_27a : \iota.\lambda V0a \in (ty\_2Eordinal\_2Eordinal\ A\_27a)$

Let  $c\_2Ewellorder\_2Ewellorder\_REP : \iota \Rightarrow \iota$  be given. Assume the following.

$$A\_27a \in ((2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})^{(ty\_2Ewellorder\_2Ewellorder\ A\_27a)})$$
 (10)

**Definition 10** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E\_7E$

**Definition 11** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.(\lambda V1f \in (2^{A\_27a}).(ap\ V1f\ V0x))$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$$
 (11)

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$$
 (12)

**Definition 12** We define  $c\_2Epair\_2EUNCURRY$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0f \in ((A\_27c)^{A\_27a \times A\_27b})$

**Definition 13** We define  $c\_Eset\_relation\_Estrict$  to be  $\lambda A\_27a : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$

**Definition 14** We define  $c\_Ewellorder\_2Eiseg$  to be  $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 15** We define  $c\_Eset\_relation\_2Erestrict$  to be  $\lambda A\_27a : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$

Let  $c\_2Ewellorder\_2Ewellorder\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ewellorder\_2Ewellorder\_ABS \\ & A\_27a \in ((ty\_2Ewellorder\_2Ewellorder\ A\_27a)^{(2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})}) \end{aligned} \quad (13)$$

**Definition 16** We define  $c\_2Ewellorder\_2Ewobound$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1w \in (ty\_2Ewellorder\ A\_27a)$

**Definition 17** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (c\_2Emin\_2E\_40\ A\_27a)\ P))))$

**Definition 18** We define  $c\_Eset\_relation\_2Erange$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$

**Definition 19** We define  $c\_Eset\_relation\_2Edomain$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$

**Definition 20** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ t1\ t2)))$

**Definition 21** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2E\_22\ A\_27a)\ s\ t))$

**Definition 22** We define  $c\_2Ewellorder\_2EelsOf$  to be  $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 23** We define  $c\_2Ewellorder\_2Eorderiso$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0w1 \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 24** We define  $c\_2Ewellorder\_2Eorderlt$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0w1 \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 25** We define  $c\_2Eordinal\_2Eordlt$  to be  $\lambda A\_27a : \iota.\lambda V0T1 \in (ty\_2Eordinal\_2Eordinal\ A\_27a)$

**Definition 26** We define  $c\_2Eordinal\_2Epreds$  to be  $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Eordinal\_2Eordinal\ A\_27a)$

**Definition 27** We define  $c\_2Epred\_set\_2EIMAGE$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1s \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2E\_27\ A\_27a)\ f\ s))$

**Definition 28** We define  $c\_2Epred\_set\_2EBIGUNION$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{(2^{A\_27a})}).(ap\ (c\_2Epred\_set\_2E\_28\ A\_27a)\ P))$

**Definition 29** We define  $c\_2Eordinal\_2Eoleast$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{(ty\_2Eordinal\_2Eordinal\ A\_27a)}).$

**Definition 30** We define  $c\_2Eordinal\_2Esup$  to be  $\lambda A\_27a : \iota.\lambda V0ordset \in (2^{(ty\_2Eordinal\_2Eordinal\ A\_27a)}).$

**Definition 31** We define  $c\_2Eordinal\_2Eomega$  to be  $\lambda A\_27a : \iota.(ap\ (c\_2Eordinal\_2Esup\ A\_27a)\ (ap\ (c\_2Epred\_set\_2E\_29\ A\_27a)\ (c\_2Epred\_set\_2E\_30\ A\_27a))))$

Let  $c\_2Eordinal\_2EordADD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eordinal\_2EordADD\ A\_27a \in (( \\ & (ty\_2Eordinal\_2Eordinal\ A\_27a)^{(ty\_2Eordinal\_2Eordinal\ A\_27a)})^{(ty\_2Eordinal\_2Eordinal\ A\_27a)}) \end{aligned} \quad (14)$$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (15)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (16)$$



Assume the following.

$$True \quad (21)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (22)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (23)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A\_27a.(p V0t)) \Leftrightarrow (p V0t))) \quad (24)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (25)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((p V0t) \Rightarrow False) \Leftrightarrow \neg(p V0t)))))) \quad (26)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t)) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True)))) \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (28)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (29)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow \neg(p V0t)) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow \neg(p V0t)))))) \quad (30)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow (p V1t2) \Rightarrow (p V2t3)) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (31)$$

Assume the following.

$$2. (((((p V0x) \Leftrightarrow (p V1x_{.27})) \wedge ((p V1x_{.27}) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_{.27})))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_{.27}) \Rightarrow (p V3y_{.27})))))) \quad (32)$$

Assume the following.

$$\forall A_{.27a}. \text{nonempty } A_{.27a} \Rightarrow (\forall V0n \in \text{ty\_2Enum\_2Enum}. (\neg((\text{ap } (c\_2Eordinal\_2EfromNat } A_{.27a}) V0n) = (c\_2Eordinal\_2Eomega } A_{.27a})))) \quad (33)$$

Assume the following.

$$\forall A_{.27a}. \text{nonempty } A_{.27a} \Rightarrow ((\text{ap } (c\_2Eordinal\_2Eomax } A_{.27a}) (\text{ap } (c\_2Eordinal\_2Epreds } A_{.27a}) (c\_2Eordinal\_2Eomega } A_{.27a}))) = (c\_2Eoption\_2ENONE (\text{ty\_2Eordinal\_2Eordinal } A_{.27a})) \quad (34)$$

Assume the following.

$$\forall A_{.27a}. \text{nonempty } A_{.27a} \Rightarrow (\forall V0b \in (\text{ty\_2Eordinal\_2Eordinal } A_{.27a}). (\forall V1a \in (\text{ty\_2Eordinal\_2Eordinal } A_{.27a}). (\forall V2c \in (\text{ty\_2Eordinal\_2Eordinal } A_{.27a}). ((p (\text{ap } (\text{ap } (c\_2Eordinal\_2Eordlt } A_{.27a}) (\text{ap } (\text{ap } (c\_2Eordinal\_2EordADD } A_{.27a}) V2c) V1a)) (\text{ap } (\text{ap } (c\_2Eordinal\_2EordADD } A_{.27a}) V2c) V0b))) \Leftrightarrow (p (\text{ap } (\text{ap } (c\_2Eordinal\_2Eordlt } A_{.27a}) V1a) V0b)))))) \quad (35)$$

Assume the following.

$$\forall A_{.27a}. \text{nonempty } A_{.27a} \Rightarrow (\forall V0a \in (\text{ty\_2Eordinal\_2Eordinal } A_{.27a}). (\forall V1s \in (2^{(\text{ty\_2Eordinal\_2Eordinal } A_{.27a})}). ((p (\text{ap } (\text{ap } (c\_2Ecardinal\_2Ecardleq } (\text{ty\_2Eordinal\_2Eordinal } A_{.27a}) V1s) (c\_2Epred\_set\_2EUNIV (\text{ty\_2Esum\_2Esum } \text{ty\_2Enum\_2Enum } A_{.27a})))) \wedge (\neg(V1s = (c\_2Epred\_set\_2EEMPTY (\text{ty\_2Eordinal\_2Eordinal } A_{.27a})))))) \Rightarrow ((\text{ap } (\text{ap } (c\_2Eordinal\_2EordADD } A_{.27a}) V0a) (\text{ap } (c\_2Eordinal\_2Esup } A_{.27a}) V1s)) = (\text{ap } (c\_2Eordinal\_2Esup } A_{.27a}) (\text{ap } (\text{ap } (c\_2Epred\_set\_2EIMAGE } (\text{ty\_2Eordinal\_2Eordinal } A_{.27a}) (\text{ap } (c\_2Eordinal\_2EordADD } A_{.27a}) V0a)) V1s)))))) \quad (36)$$

Assume the following.

$$\begin{aligned}
& \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow (\forall V0f \in ((ty\_2Eordinal\_2Eordinal \\
& A_{.27a})^{(ty\_2Eordinal\_2Eordinal\ A_{.27a})}). (\forall V1a \in (ty\_2Eordinal\_2Eordinal \\
& A_{.27a}). ((\forall V2s \in (2^{(ty\_2Eordinal\_2Eordinal\ A_{.27a})}). \\
& ((p\ (ap\ (ap\ (c\_2Ecardinal\_2Ecardleq\ (ty\_2Eordinal\_2Eordinal \\
& A_{.27a})\ (ty\_2Esum\_2Esum\ ty\_2Enum\_2Enum\ A_{.27a}))\ V2s)\ (c\_2Epred\_set\_2EUNIV \\
& (ty\_2Esum\_2Esum\ ty\_2Enum\_2Enum\ A_{.27a})))) \wedge (\neg(V2s = (c\_2Epred\_set\_2EEMPTY \\
& (ty\_2Eordinal\_2Eordinal\ A_{.27a})))))) \Rightarrow ((ap\ V0f\ (ap\ (c\_2Eordinal\_2Esup \\
& A_{.27a})\ V2s)) = (ap\ (c\_2Eordinal\_2Esup\ A_{.27a})\ (ap\ (ap\ (c\_2Epred\_set\_2EIMAGE \\
& (ty\_2Eordinal\_2Eordinal\ A_{.27a})\ (ty\_2Eordinal\_2Eordinal\ A_{.27a})) \\
& V0f)\ V2s)))) \wedge ((\forall V3x \in (ty\_2Eordinal\_2Eordinal\ A_{.27a}). \\
& (\forall V4y \in (ty\_2Eordinal\_2Eordinal\ A_{.27a}). ((p\ (ap\ (ap\ (c\_2Eordinal\_2Eordlt \\
& A_{.27a})\ V3x)\ V4y)) \Rightarrow (p\ (ap\ (ap\ (c\_2Eordinal\_2Eordlt\ A_{.27a})\ (ap\ V0f \\
& V3x))\ (ap\ V0f\ V4y)))))) \wedge (((ap\ (c\_2Eordinal\_2Eomax\ A_{.27a})\ (ap\ ( \\
& c\_2Eordinal\_2Epreds\ A_{.27a})\ V1a)) = (c\_2Eoption\_2ENONE\ (ty\_2Eordinal\_2Eordinal \\
& A_{.27a}))) \wedge (\neg(V1a = (ap\ (c\_2Eordinal\_2EfromNat\ A_{.27a})\ c\_2Enum\_2E0)))))) \Rightarrow \\
& ((ap\ (c\_2Eordinal\_2Eomax\ A_{.27a})\ (ap\ (c\_2Eordinal\_2Epreds\ A_{.27a}) \\
& (ap\ V0f\ V1a))) = (c\_2Eoption\_2ENONE\ (ty\_2Eordinal\_2Eordinal\ A_{.27a})))))) \\
& \hspace{15em} (37)
\end{aligned}$$

**Theorem 1**

$$\begin{aligned}
& \forall A_{.27a}.nonempty\ A_{.27a} \Rightarrow (\forall V0a \in (ty\_2Eordinal\_2Eordinal \\
& A_{.27a}). ((ap\ (c\_2Eordinal\_2Eomax\ A_{.27a})\ (ap\ (c\_2Eordinal\_2Epreds \\
& A_{.27a})\ (ap\ (ap\ (c\_2Eordinal\_2EordADD\ A_{.27a})\ V0a)\ (c\_2Eordinal\_2Eomega \\
& A_{.27a})))) = (c\_2Eoption\_2ENONE\ (ty\_2Eordinal\_2Eordinal\ A_{.27a}))))))
\end{aligned}$$