

# thm\_2Eordinal\_2Ecsup\_lesup (TMZA- zUHTp5P8SSPPXNHxGGP21Scxxz5uvy4)

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**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. \text{inj\_o } (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2ET` to be  $(\text{ap } (\text{ap } (\text{c\_2Emin\_2E\_3D } (2^2))) (\lambda V0x \in 2. V0x)) (\lambda V1x \in 2. V1x)$

**Definition 3** We define `c_2Ebool_2EIN` to be  $\lambda A. \lambda a : \iota. (\lambda V0x \in A. \lambda V1f \in (2^{A-27a}). (\text{ap } V1f V0x))$

**Definition 4** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. P x) \text{ then } (\lambda x. x \in A \wedge P x)$  of type  $\iota \Rightarrow \iota$ .

**Definition 5** We define `c_2Ebool_2E_3F` to be  $\lambda A. \lambda a : \iota. (\lambda V0P \in (2^{A-27a}). (\text{ap } V0P (\text{ap } (\text{c\_2Emin\_2E\_40 } A))))$

Let `ty_2Enum_2Enum` :  $\iota$  be given. Assume the following.

$$\text{nonempty } \text{ty\_2Enum\_2Enum} \tag{1}$$

Let `ty_2Esum_2Esum` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \forall A1. \text{nonempty } A1 \Rightarrow \text{nonempty } (\text{ty\_2Esum\_2Esum } A0 A1) \tag{2}$$

Let `ty_2Ewellorder_2Ewellorder` :  $\iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \text{nonempty } (\text{ty\_2Ewellorder\_2Ewellorder } A0) \tag{3}$$

Let `ty_2Eordinal_2Eordinal` :  $\iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. \text{nonempty } A0 \Rightarrow \text{nonempty } (\text{ty\_2Eordinal\_2Eordinal } A0) \tag{4}$$

Let `c_2Eordinal_2Eordinal__REP__CLASS` :  $\iota \Rightarrow \iota$  be given. Assume the following.

$$A. \lambda a : \iota. (\text{nonempty } A \Rightarrow \text{nonempty } (\text{c\_2Eordinal\_2Eordinal\_REP\_CLASS } A a)) \tag{5}$$

**Definition 6** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota. (\lambda V0P \in (2^{A\_27a}). (ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a})))$

**Definition 7** We define  $c\_2Eordinal\_2Eordinal\_REP$  to be  $\lambda A\_27a : \iota. \lambda V0a \in (ty\_2Eordinal\_2Eordinal A\_27a)$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \quad (6)$$

Let  $c\_2Ewellorder\_2Ewellorder\_REP : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ewellorder\_2Ewellorder\_REP A\_27a \in ((2^{(ty\_2Epair\_2Eprod A\_27a A\_27a)})^{(ty\_2Ewellorder\_2Ewellorder A\_27a)}) \quad (7)$$

**Definition 8** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 9** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 10** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2. (ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2F))$

**Definition 11** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2. (ap (c\_2Emin\_2E\_3D\_3D\_3E V2t) c\_2Ebool\_2E\_2F)))))$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EABS\_prod A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (8)$$

**Definition 12** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1y \in A\_27b. (ap (c\_2Emin\_2E\_3D\_3D\_3E (c\_2Epair\_2EABS\_prod A\_27a A\_27b) V0x) V1y))$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2ESND A\_27a A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \quad (9)$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EFST A\_27a A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \quad (10)$$

**Definition 13** We define  $c\_2Epair\_2EUNCURRY$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda A\_27c : \iota. \lambda V0f \in ((A\_27c)^{A\_27a})$

Let  $c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC A\_27a A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod A\_27a 2)^{A\_27b}}) \quad (11)$$

**Definition 14** We define  $c\_Eset\_relation\_Estrict$  to be  $\lambda A\_27a : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$ ,

**Definition 15** We define  $c\_Ewellorder\_Eiseg$  to be  $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$ ,

**Definition 16** We define  $c\_Eset\_relation\_Erestrict$  to be  $\lambda A\_27a : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$ ,

Let  $c\_2Ewellorder\_2Ewellorder\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ewellorder\_2Ewellorder\_ABS \\ & A\_27a \in ((ty\_2Ewellorder\_2Ewellorder\ A\_27a)^{(2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})}) \end{aligned} \quad (12)$$

**Definition 17** We define  $c\_2Ewellorder\_2Ewobound$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1w \in (ty\_2Ewellorder\ A\_27a)$ ,

**Definition 18** We define  $c\_Eset\_relation\_Erange$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$ ,

**Definition 19** We define  $c\_Eset\_relation\_Edomain$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)})$ ,

**Definition 20** We define  $c\_Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2))))$ ,

**Definition 21** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).\lambda V1t \in (2^{A\_27a}).(ap\ (c\_2Epred\_set\_2E\_21\ 2)\ s))$ ,

**Definition 22** We define  $c\_2Ewellorder\_2EelsOf$  to be  $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$ ,

**Definition 23** We define  $c\_2Ewellorder\_2Eorderiso$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0w1 \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$ ,

**Definition 24** We define  $c\_2Ewellorder\_2Eorderlt$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0w1 \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$ ,

**Definition 25** We define  $c\_2Eordinal\_2Eordlt$  to be  $\lambda A\_27a : \iota.\lambda V0T1 \in (ty\_2Eordinal\_2Eordinal\ A\_27a).$

**Definition 26** We define  $c\_2Eordinal\_2Epreds$  to be  $\lambda A\_27a : \iota.\lambda V0w \in (ty\_2Eordinal\_2Eordinal\ A\_27a).$

**Definition 27** We define  $c\_2Epred\_set\_2EIMAGE$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1s \in (2^{A\_27b})$ ,

**Definition 28** We define  $c\_2Epred\_set\_2EBIGUNION$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{(2^{A\_27a})}).(ap\ (c\_2Epred\_set\_2E\_21\ 2)\ P))$ ,

**Definition 29** We define  $c\_2Eordinal\_2Eoleast$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{(ty\_2Eordinal\_2Eordinal\ A\_27a)}).$

**Definition 30** We define  $c\_2Eordinal\_2Esup$  to be  $\lambda A\_27a : \iota.\lambda V0ordset \in (2^{(ty\_2Eordinal\_2Eordinal\ A\_27a)}).$

**Definition 31** We define  $c\_2Epred\_set\_2EUNIV$  to be  $\lambda A\_27a : \iota.(\lambda V0x \in A\_27a.c\_2Ebool\_2EET)$ .

**Definition 32** We define  $c\_2Epred\_set\_2EINJ$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1s \in (2^{A\_27b})$ ,

**Definition 33** We define  $c\_2Epred\_set\_2Ecountable$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{A\_27a}).(ap\ (c\_2Ebool\_2E\_3F\ 2)\ s))$ ,

Assume the following.

$$True \quad (13)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (14)$$

Assume the following.

$$(\forall V0t \in 2. (False \Rightarrow (p V0t))) \quad (15)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow \neg(p V0t)) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow \neg(p V0t)))))) \quad (16)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in (2^{A.27a}). (\neg(\exists V1x \in A.27a.(p (ap V0P V1x)))) \Leftrightarrow (\forall V2x \in A.27a. (\neg(p (ap V0P V2x))))) \quad (17)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \wedge (p V1B))) \Leftrightarrow ((\neg(p V0A) \vee \neg(p V1B)))) \wedge (((\neg(p V0A) \vee (p V1B)) \Leftrightarrow ((\neg(p V0A) \wedge \neg(p V1B))))))))) \quad (18)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0w \in (ty.2Eordinal.2Eordinal A.27a). (\neg(p (ap (ap (c.2Eordinal.2Eordlt A.27a) V0w) V0w)))) \quad (19)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0s \in (2^{(ty.2Eordinal.2Eordinal A.27a)}). ((p (ap (c.2Epred_set.2Ecountable (ty.2Eordinal.2Eordinal A.27a) V0s)) \Rightarrow (\forall V1b \in (ty.2Eordinal.2Eordinal A.27a). ((p (ap (ap (c.2Eordinal.2Eordlt A.27a) V1b) (ap (c.2Eordinal.2Esup A.27a) V0s))) \Leftrightarrow (\exists V2d \in (ty.2Eordinal.2Eordinal A.27a). ((p (ap (ap (c.2Ebool.2EIN (ty.2Eordinal.2Eordinal A.27a) V2d) V0s)) \wedge (p (ap (ap (c.2Eordinal.2Eordlt A.27a) V1b) V2d)))))))))) \quad (20)$$

Assume the following.

$$(\forall V0t \in 2. ((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \quad (21)$$

Assume the following.

$$(\forall V0A \in 2. ((p V0A) \Rightarrow ((\neg(p V0A)) \Rightarrow False))) \quad (22)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg((p V0A) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow False) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (23)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (((\neg(\neg(p V0A)) \vee (p V1B))) \Rightarrow False) \Leftrightarrow ((p V0A) \Rightarrow ((\neg(p V1B)) \Rightarrow False)))) \quad (24)$$

Assume the following.

$$(\forall V0A \in 2. (((\neg(p V0A)) \Rightarrow False) \Rightarrow (((p V0A) \Rightarrow False) \Rightarrow False))) \quad (25)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Leftrightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee ((p V1q) \vee (p V2r))) \wedge (((p V0p) \vee (\neg(p V2r)) \vee (\neg(p V1q)))) \wedge (((p V1q) \vee (\neg(p V2r)) \vee (\neg(p V0p)))) \wedge ((p V2r) \vee ((\neg(p V1q)) \vee (\neg(p V0p)))))))))) \quad (26)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \vee (p V2r)) \Leftrightarrow (((p V0p) \vee (\neg(p V1q))) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((p V1q) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \quad (27)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Rightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee (\neg(p V2r))) \wedge ((\neg(p V1q)) \vee ((p V2r) \vee (\neg(p V0p)))))))))) \quad (28)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow (\neg(p V1q))) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge ((\neg(p V1q)) \vee (\neg(p V0p)))))) \quad (29)$$

### Theorem 1

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0s \in (2^{(ty.2Eordinal.2Eordinal A.27a)}). ((p (ap (c.2Epred\_set.2Ecountable (ty.2Eordinal.2Eordinal A.27a)) V0s)) \Rightarrow (\forall V1d \in (ty.2Eordinal.2Eordinal A.27a). ((p (ap (ap (c.2Ebool.2EIN (ty.2Eordinal.2Eordinal A.27a)) V1d) V0s)) \Rightarrow (\neg(p (ap (ap (c.2Eordinal.2Eordlt A.27a) (ap (c.2Eordinal.2Esup A.27a) V0s)) V1d)))))))$$