

# thm\_2Eordinal\_2Emul\_\_omega\_\_islimit (TM- PRUyN4HftSaXTn1zmJ9wEZt9ZUKYAawT9o)

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**Definition 1** We define `c_2Emin_2E_3D` to be  $\lambda A. \lambda x \in A. \lambda y \in A. inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define `c_2Ebool_2E_2T` to be  $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define `c_2Ebool_2E_21` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (ap (ap (c_2Emin_2E_3D (2^{A-27a})) (\lambda V1P \in 2.V1P)) (\lambda V2P \in 2.V2P)))$

**Definition 4** We define `c_2Ebool_2E_2F` to be  $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define `c_2Epred__set_2EEMPTY` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a.c_2Ebool_2E_2F)$ .

**Definition 6** We define `c_2Emin_2E_40` to be  $\lambda A. \lambda P \in 2^A. \mathbf{if} (\exists x \in A. p (ap P x)) \mathbf{then} (the (\lambda x. x \in A \wedge p (ap P x)))$  of type  $\iota \Rightarrow \iota$ .

**Definition 7** We define `c_2Ebool_2E_3F` to be  $\lambda A. 27a : \iota. (\lambda V0P \in (2^{A-27a}). (ap V0P (ap (c_2Emin_2E_40 A) (\lambda V1P \in 2.V1P))))$

**Definition 8** We define `c_2Emin_2E_3D_3D_3E` to be  $\lambda P \in 2. \lambda Q \in 2. inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 9** We define `c_2Ebool_2E_5C_2F` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))))$

**Definition 10** We define `c_2Ebool_2E_2IN` to be  $\lambda A. 27a : \iota. (\lambda V0x \in A. 27a. (\lambda V1f \in (2^{A-27a}). (ap V1f V0x)))$

**Definition 11** We define `c_2Ebool_2E_2F_5C` to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t))))$

Let `ty_2Epair_2Eprod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow \forall A1. nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let `c_2Epair_2EABS__prod` :  $\iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A. 27a. nonempty A. 27a \Rightarrow \forall A. 27b. nonempty A. 27b \Rightarrow c\_2Epair\_2EABS\_prod A. 27a A. 27b \in ((ty\_2Epair\_2Eprod A. 27a A. 27b)^{(2^{A-27b})^{A-27a}}) \tag{2}$$

**Definition 12** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1y \in A\_27b. (ap (c\_2Epred\_set\_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota)$  be given. Assume the following.

$$\forall A\_27a. nonempty\ A\_27a \Rightarrow \forall A\_27b. nonempty\ A\_27b \Rightarrow c\_2Epred\_set\_2EGSPEC\ A\_27a\ A\_27b \in ((2^{A\_27a})^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \quad (3)$$

**Definition 13** We define  $c\_2Epred\_set\_2EIMAGE$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0f \in (A\_27b^{A\_27a}). \lambda V1s \in$  Let  $ty\_2Eenum\_2Eenum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eenum\_2Eenum \quad (4)$$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty\ A0 \Rightarrow \forall A1. nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \quad (5)$$

Let  $ty\_2Ewellorder\_2Ewellorder : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ewellorder\_2Ewellorder\ A0) \quad (6)$$

Let  $ty\_2Eordinal\_2Eordinal : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty\ A0 \Rightarrow nonempty\ (ty\_2Eordinal\_2Eordinal\ A0) \quad (7)$$

Let  $c\_2Eordinal\_2Eordinal\_REP\_CLASS : \iota \Rightarrow \iota$  be given. Assume the following.

$$A\_27a \in ((2^{(ty\_2Ewellorder\_2Ewellorder\ (ty\_2Esum\_2Esum\ ty\_2Eenum\_2Eenum\ A\_27a))})^{(ty\_2Eordinal\_2Eordinal\ A\_27a)}) \quad (8)$$

**Definition 14** We define  $c\_2Eordinal\_2Eordinal\_REP$  to be  $\lambda A\_27a : \iota. \lambda V0a \in (ty\_2Eordinal\_2Eordinal\ A\_27a)$  Let  $c\_2Ewellorder\_2Ewellorder\_REP : \iota \Rightarrow \iota$  be given. Assume the following.

$$A\_27a \in ((2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})^{(ty\_2Ewellorder\_2Ewellorder\_REP\ A\_27a)}) \quad (9)$$

**Definition 15** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2. (ap (ap\ c\_2Emin\_2E\_3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E\_7E))$  Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty\ A\_27a \Rightarrow \forall A\_27b. nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \quad (10)$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty\ A\_27a \Rightarrow \forall A\_27b. nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \quad (11)$$

**Definition 16** We define  $c\_2Epair\_2EUNCURRY$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda A\_27c : \iota. \lambda V0f \in ((A\_27c^{A\_27a})$

**Definition 17** We define  $c\_2Eset\_relation\_2Estrict$  to be  $\lambda A\_27a : \iota. \lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$

**Definition 18** We define  $c\_2Ewellorder\_2Eiseg$  to be  $\lambda A\_27a : \iota. \lambda V0w \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 19** We define  $c\_2Eset\_relation\_2Erestrict$  to be  $\lambda A\_27a : \iota. \lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$

Let  $c\_2Ewellorder\_2Ewellorder\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a. nonempty\ A\_27a \Rightarrow c\_2Ewellorder\_2Ewellorder\_ABS \\ & A\_27a \in ((ty\_2Ewellorder\_2Ewellorder\ A\_27a)^{(2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})}) \end{aligned} \quad (12)$$

**Definition 20** We define  $c\_2Ewellorder\_2Ewobound$  to be  $\lambda A\_27a : \iota. \lambda V0x \in A\_27a. \lambda V1w \in (ty\_2Ewellorder\ A\_27a)$

**Definition 21** We define  $c\_2Eset\_relation\_2Erange$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$

**Definition 22** We define  $c\_2Eset\_relation\_2Edomain$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0r \in (2^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27a)})$

**Definition 23** We define  $c\_2Epred\_set\_2EUNION$  to be  $\lambda A\_27a : \iota. \lambda V0s \in (2^{A\_27a}). \lambda V1t \in (2^{A\_27a}). (ap\ (c\_2Epred\_set\ A\_27a)\ s)$

**Definition 24** We define  $c\_2Ewellorder\_2EelsOf$  to be  $\lambda A\_27a : \iota. \lambda V0w \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 25** We define  $c\_2Ewellorder\_2Eorderiso$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0w1 \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 26** We define  $c\_2Ewellorder\_2Eorderlt$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0w1 \in (ty\_2Ewellorder\_2Ewellorder\ A\_27a)$

**Definition 27** We define  $c\_2Eordinal\_2Eordlt$  to be  $\lambda A\_27a : \iota. \lambda V0T1 \in (ty\_2Eordinal\_2Eordinal\ A\_27a)$

**Definition 28** We define  $c\_2Eordinal\_2Epreds$  to be  $\lambda A\_27a : \iota. \lambda V0w \in (ty\_2Eordinal\_2Eordinal\ A\_27a)$

**Definition 29** We define  $c\_2Epred\_set\_2EBIGUNION$  to be  $\lambda A\_27a : \iota. \lambda V0P \in (2^{(2^{A\_27a})}). (ap\ (c\_2Epred\_set\ A\_27a)\ P)$

**Definition 30** We define  $c\_2Eordinal\_2Eoleast$  to be  $\lambda A\_27a : \iota. \lambda V0P \in (2^{(ty\_2Eordinal\_2Eordinal\ A\_27a)})$

**Definition 31** We define  $c\_2Eordinal\_2Esup$  to be  $\lambda A\_27a : \iota. \lambda V0ordset \in (2^{(ty\_2Eordinal\_2Eordinal\ A\_27a)})$

**Definition 32** We define  $c\_2Eordinal\_2EordSUC$  to be  $\lambda A\_27a : \iota. \lambda V0a \in (ty\_2Eordinal\_2Eordinal\ A\_27a)$

Let  $c\_2Enum\_2EZZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZZERO\_REP \in \omega \quad (13)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (14)$$

**Definition 33** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZZERO\_REP)$ .

Let  $c\_2Eordinal\_2EfromNat : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eordinal\_2EfromNat\ A\_27a \in ( (ty\_2Eordinal\_2Eordinal\ A\_27a)^{ty\_2Eenum\_2Eenum} ) \quad (15)$$

**Definition 34** We define  $c\_2Eordinal\_2Eomega$  to be  $\lambda A\_27a : \iota.(ap\ (c\_2Eordinal\_2Esup\ A\_27a)\ (ap\ (c\_2Epr$

Let  $c\_2Eordinal\_2EordADD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eordinal\_2EordADD\ A\_27a \in ( ( (ty\_2Eordinal\_2Eordinal\ A\_27a)^{(ty\_2Eordinal\_2Eordinal\ A\_27a)} )^{(ty\_2Eordinal\_2Eordinal\ A\_27a)} ) \quad (16)$$

Let  $c\_2Eordinal\_2EordMULT : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eordinal\_2EordMULT\ A\_27a \in ( ((ty\_2Eordinal\_2Eordinal\ A\_27a)^{(ty\_2Eordinal\_2Eordinal\ A\_27a)})^{(ty\_2Eordinal\_2Eordinal\ A\_27a)} ) \quad (17)$$

Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \quad (18)$$

**Definition 35** We define  $c\_2Eone\_2Eone$  to be  $(ap\ (c\_2Emin\_2E40\ ty\_2Eone\_2Eone)\ (\lambda V0x \in ty\_2Eone\_2Eone$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b \in ( (ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2} ) \quad (19)$$

**Definition 36** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap\ (c\_2Esum\_2EABS$

Let  $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Eoption\_2Eoption\ A0) \quad (20)$$

Let  $c\_2Eoption\_2Eoption\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eoption\_2Eoption\_ABS\ A\_27a \in ( (ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ ty\_2Eone\_2Eone)} ) \quad (21)$$

**Definition 37** We define  $c\_2Eoption\_2EENONE$  to be  $\lambda A\_27a : \iota.(ap\ (c\_2Eoption\_2Eoption\_ABS\ A\_27a)\ ($

**Definition 38** We define  $c\_2Epred\_set\_2EINSERT$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.\lambda V1s \in (2^{A\_27a}).(ap\ (c$

**Definition 39** We define  $c\_2Eset\_relation\_2Emaximal\_elements$  to be  $\lambda A\_27a : \iota.\lambda V0xs \in (2^{A\_27a}).\lambda V1r \in$

**Definition 40** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap\ (c\_2Esum\_2EABS$

**Definition 41** We define  $c\_2Eoption\_2ESOME$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.(ap\ (c\_2Eoption\_2Eoption$

**Definition 42** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

**Definition 43** We define  $c\_2Eoption\_2Esome$  to be  $\lambda A\_27a : \iota.\lambda V0P \in (2^{A\_27a}).(ap (ap (ap (c\_2Ebool\_2ECC$

**Definition 44** We define  $c\_2Eordinal\_2Eomax$  to be  $\lambda A\_27a : \iota.\lambda V0s \in (2^{(ty\_2Eordinal\_2Eordinal A\_27a)}).(ap ($

Assume the following.

$$True \quad (22)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (23)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow ((ap (c\_2Eordinal\_2Eomax A\_27a) \\ & (c\_2Epred\_set\_2EEMPTY (ty\_2Eordinal\_2Eordinal A\_27a))) = ( \\ & c\_2Eoption\_2ENONE (ty\_2Eordinal\_2Eordinal A\_27a))) \quad (24) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow ((ap (c\_2Eordinal\_2Epreds A\_27a) \\ & (ap (c\_2Eordinal\_2EfromNat A\_27a) c\_2Enum\_2E0)) = (c\_2Epred\_set\_2EEMPTY \\ & (ty\_2Eordinal\_2Eordinal A\_27a))) \quad (25) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0a \in (ty\_2Eordinal\_2Eordinal \\ & A\_27a).((V0a = (ap (c\_2Eordinal\_2EfromNat A\_27a) c\_2Enum\_2E0)) \vee \\ & ((\exists V1a0 \in (ty\_2Eordinal\_2Eordinal A\_27a).(V0a = (ap (c\_2Eordinal\_2EordSUC \\ & A\_27a) V1a0))) \vee ((p (ap (ap (c\_2Eordinal\_2Eordlt A\_27a) (ap (c\_2Eordinal\_2EfromNat \\ & A\_27a) c\_2Enum\_2E0)) V0a)) \wedge ((ap (c\_2Eordinal\_2Eomax A\_27a) ( \\ & ap (c\_2Eordinal\_2Epreds A\_27a) V0a)) = (c\_2Eoption\_2ENONE (ty\_2Eordinal\_2Eordinal \\ & A\_27a)))))))) \quad (26) \end{aligned}$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0b \in (ty\_2Eordinal\_2Eordinal \\ & A\_27a).(((ap (ap (c\_2Eordinal\_2EordMULT A\_27a) V0b) (ap (c\_2Eordinal\_2EfromNat \\ & A\_27a) c\_2Enum\_2E0)) = (ap (c\_2Eordinal\_2EfromNat A\_27a) c\_2Enum\_2E0)) \wedge \\ & ((\forall V1a \in (ty\_2Eordinal\_2Eordinal A\_27a).((ap (ap (c\_2Eordinal\_2EordMULT \\ & A\_27a) V0b) (ap (c\_2Eordinal\_2EordSUC A\_27a) V1a)) = (ap (ap (c\_2Eordinal\_2EordADD \\ & A\_27a) (ap (ap (c\_2Eordinal\_2EordMULT A\_27a) V0b) V1a)) V0b))) \wedge \\ & (\forall V2a \in (ty\_2Eordinal\_2Eordinal A\_27a).(((p (ap (ap (c\_2Eordinal\_2Eordlt \\ & A\_27a) (ap (c\_2Eordinal\_2EfromNat A\_27a) c\_2Enum\_2E0)) V2a)) \wedge \\ & ((ap (c\_2Eordinal\_2Eomax A\_27a) (ap (c\_2Eordinal\_2Epreds A\_27a) \\ & V2a)) = (c\_2Eoption\_2ENONE (ty\_2Eordinal\_2Eordinal A\_27a)))))) \Rightarrow \\ & ((ap (ap (c\_2Eordinal\_2EordMULT A\_27a) V0b) V2a) = (ap (c\_2Eordinal\_2Esup \\ & A\_27a) (ap (ap (c\_2Epred\_set\_2EIMAGE (ty\_2Eordinal\_2Eordinal \\ & A\_27a) (ty\_2Eordinal\_2Eordinal A\_27a)) (ap (c\_2Eordinal\_2EordMULT \\ & A\_27a) V0b)) (ap (c\_2Eordinal\_2Epreds A\_27a) V2a)))))) \quad (27) \end{aligned}$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0a \in (ty\_2Eordinal\_2Eordinal \\ A.27a).((ap\ (c\_2Eordinal\_2Eomax\ A.27a)\ (ap\ (c\_2Eordinal\_2Epreds \\ A.27a)\ (ap\ (ap\ (c\_2Eordinal\_2EordADD\ A.27a)\ V0a)\ (c\_2Eordinal\_2Eomega \\ A.27a)))) = (c\_2Eoption\_2ENONE\ (ty\_2Eordinal\_2Eordinal\ A.27a)))) \end{aligned} \quad (28)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0b \in (ty\_2Eordinal\_2Eordinal \\ A.27a).(\forall V1a \in (ty\_2Eordinal\_2Eordinal\ A.27a).((ap\ ( \\ c\_2Eordinal\_2Eomax\ A.27a)\ (ap\ (c\_2Eordinal\_2Epreds\ A.27a)\ V1a)) = \\ (c\_2Eoption\_2ENONE\ (ty\_2Eordinal\_2Eordinal\ A.27a)))) \Rightarrow ((ap\ ( \\ c\_2Eordinal\_2Eomax\ A.27a)\ (ap\ (c\_2Eordinal\_2Epreds\ A.27a)\ (ap \\ (ap\ (c\_2Eordinal\_2EordMULT\ A.27a)\ V0b)\ V1a)))) = (c\_2Eoption\_2ENONE \\ (ty\_2Eordinal\_2Eordinal\ A.27a)))) \end{aligned} \quad (29)$$

**Theorem 1**

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0a \in (ty\_2Eordinal\_2Eordinal \\ A.27a).((ap\ (c\_2Eordinal\_2Eomax\ A.27a)\ (ap\ (c\_2Eordinal\_2Epreds \\ A.27a)\ (ap\ (ap\ (c\_2Eordinal\_2EordMULT\ A.27a)\ (c\_2Eordinal\_2Eomega \\ A.27a)\ V0a)))) = (c\_2Eoption\_2ENONE\ (ty\_2Eordinal\_2Eordinal \\ A.27a)))) \end{aligned}$$