

thm_2Eordinal_2EordADD_0L
(TMU4HnxyS1QzayPArd7KrMvua7C4jrALRd2)

October 26, 2020

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 4 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 5 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 6 We define $c_2Ebool_2E_27E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F$

Let $ty_2Eenum_2Eenum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eenum_2Eenum \tag{1}$$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \tag{2}$$

Let $ty_2Ewellorder_2Ewellorder : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Ewellorder_2Ewellorder\ A0) \tag{3}$$

Let $ty_2Eordinal_2Eordinal : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Eordinal_2Eordinal\ A0) \tag{4}$$

Let $c_2Eordinal_2Eordinal_REP_CLASS : \iota \Rightarrow \iota$ be given. Assume the following.

$$A_27a \in ((2^{(ty_2Ewellorder_2Ewellorder\ (ty_2Esum_2Esum\ ty_2Eenum_2Eenum\ A_27a))})^{(ty_2Eordinal_2Eordinal\ A_27a)}) \tag{5}$$

Definition 7 We define $c_2Emin_2E.40$ to be $\lambda A.\lambda P \in 2^A$. **if** $(\exists x \in A.p (ap P x))$ **then** (the $(\lambda x.x \in A \wedge p$ of type $\iota \Rightarrow \iota$).

Definition 8 We define $c_2Eordinal_2Eordinal_REP$ to be $\lambda A.27a : \iota.\lambda V0a \in (ty_2Eordinal_2Eordinal A.27a$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \quad (6)$$

Let $c_2Ewellorder_2Ewellorder_REP : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow c_2Ewellorder_2Ewellorder_REP A.27a \in ((2^{(ty_2Epair_2Eprod A.27a A.27a)})^{(ty_2Ewellorder_2Ewellorder A.27a)}) \quad (7)$$

Definition 9 We define $c_2Ebool_2E2F.5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E21 2) (\lambda V2t \in 2$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow \forall A.27b.nonempty A.27b \Rightarrow c_2Epair_2EABS_prod A.27a A.27b \in ((ty_2Epair_2Eprod A.27a A.27b)^{(2^{A.27b})^{A.27a}}) \quad (8)$$

Definition 10 We define c_2Epair_2E2C to be $\lambda A.27a : \iota.\lambda A.27b : \iota.\lambda V0x \in A.27a.\lambda V1y \in A.27b.(ap (c_2$

Definition 11 We define c_2Ebool_2EIN to be $\lambda A.27a : \iota.(\lambda V0x \in A.27a.(\lambda V1f \in (2^{A.27a}).(ap V1f V0x))$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow \forall A.27b.nonempty A.27b \Rightarrow c_2Epair_2ESND A.27a A.27b \in (A.27b^{(ty_2Epair_2Eprod A.27a A.27b)}) \quad (9)$$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow \forall A.27b.nonempty A.27b \Rightarrow c_2Epair_2EFST A.27a A.27b \in (A.27a^{(ty_2Epair_2Eprod A.27a A.27b)}) \quad (10)$$

Definition 12 We define $c_2Epair_2EUNCURRY$ to be $\lambda A.27a : \iota.\lambda A.27b : \iota.\lambda A.27c : \iota.\lambda V0f \in ((A.27c^{A.27$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow \forall A.27b.nonempty A.27b \Rightarrow c_2Epred_set_2EGSPEC A.27a A.27b \in ((2^{A.27a})^{(ty_2Epair_2Eprod A.27a 2)^{A.27b}}) \quad (11)$$

Definition 13 We define $c_2Eset_relation_2Estrict$ to be $\lambda A.27a : \iota.\lambda V0r \in (2^{(ty_2Epair_2Eprod A.27a A.27a)})$

Definition 14 We define $c_2Ewellorder_2Eiseg$ to be $\lambda A.27a : \iota.\lambda V0w \in (ty_2Ewellorder_2Ewellorder A.27a$

Definition 15 We define $c_Eset_relation_Erestrict$ to be $\lambda A_27a : \iota. \lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})$

Let $c_Ewellorder_Ewellorder_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} & \forall A_27a. nonempty\ A_27a \Rightarrow c_Ewellorder_Ewellorder_ABS \\ & A_27a \in ((ty_2Ewellorder_Ewellorder\ A_27a)^{(2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})}) \end{aligned} \quad (12)$$

Definition 16 We define $c_Ewellorder_Ewobound$ to be $\lambda A_27a : \iota. \lambda V0x \in A_27a. \lambda V1w \in (ty_2Ewellorder\ A_27a)$

Definition 17 We define $c_Ebool_2E_3F$ to be $\lambda A_27a : \iota. (\lambda V0P \in (2^{A_27a}). (ap\ V0P\ (ap\ (c_2Emin_2E_40\ A_27a)\ P)))$

Definition 18 We define $c_Eset_relation_Erange$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27b)})$

Definition 19 We define $c_Eset_relation_Edomain$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27b)})$

Definition 20 We define $c_Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c_2Ebool_2E_21\ 2)\ t1\ t2)))$

Definition 21 We define $c_Epred_set_2EUNION$ to be $\lambda A_27a : \iota. \lambda V0s \in (2^{A_27a}). \lambda V1t \in (2^{A_27a}). (ap\ (c_2Epred_set_2E_22\ A_27a)\ s\ t)$

Definition 22 We define $c_Ewellorder_2EelsOf$ to be $\lambda A_27a : \iota. \lambda V0w \in (ty_2Ewellorder_Ewellorder\ A_27a)$

Definition 23 We define $c_Ewellorder_2Eorderiso$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0w1 \in (ty_2Ewellorder_Ewellorder\ A_27a)$

Definition 24 We define $c_Ewellorder_2Eorderlt$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0w1 \in (ty_2Ewellorder_Ewellorder\ A_27a)$

Definition 25 We define $c_2Eordinal_2Eordlt$ to be $\lambda A_27a : \iota. \lambda V0T1 \in (ty_2Eordinal_2Eordinal\ A_27a)$

Definition 26 We define $c_2Eordinal_2Epreds$ to be $\lambda A_27a : \iota. \lambda V0w \in (ty_2Eordinal_2Eordinal\ A_27a)$

Definition 27 We define $c_2Epred_set_2EIMAGE$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0f \in (A_27b^{A_27a}). \lambda V1s \in (2^{A_27a})$

Definition 28 We define $c_2Epred_set_2EBIGUNION$ to be $\lambda A_27a : \iota. \lambda V0P \in (2^{(2^{A_27a})}). (ap\ (c_2Epred_set_2E_28\ A_27a)\ P)$

Definition 29 We define $c_2Eordinal_2Eoleast$ to be $\lambda A_27a : \iota. \lambda V0P \in (2^{(ty_2Eordinal_2Eordinal\ A_27a)})$

Definition 30 We define $c_2Eordinal_2Esup$ to be $\lambda A_27a : \iota. \lambda V0ordset \in (2^{(ty_2Eordinal_2Eordinal\ A_27a)})$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \quad (13)$$

Definition 31 We define c_2Eone_2Eone to be $(ap\ (c_2Emin_2E_40\ ty_2Eone_2Eone)\ (\lambda V0x \in ty_2Eone_2Eone))$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} & \forall A_27a. nonempty\ A_27a \Rightarrow \forall A_27b. nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum \\ & A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \end{aligned} \quad (14)$$

Definition 32 We define c_2Esum_2EINR to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0e \in A_27b. (ap\ (c_2Esum_2EABS_sum\ A_27a)\ e)$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. ((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (22)$$

Assume the following.

$$\forall A_27a. \text{nonempty } A_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A_27a. (p V0t)) \Leftrightarrow (p V0t))) \quad (23)$$

Assume the following.

$$(\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (24)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (25)$$

Assume the following.

$$\forall A_27a. \text{nonempty } A_27a \Rightarrow (\forall V0x \in A_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (26)$$

Assume the following.

$$\forall A_27a. \text{nonempty } A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (27)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t)))))) \quad (28)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \quad (29)$$

Assume the following.

$$(\forall V0x \in 2. (\forall V1x_27 \in 2. (\forall V2y \in 2. (\forall V3y_27 \in 2. (((p V0x) \Leftrightarrow (p V1x_27)) \wedge ((p V1x_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_27)))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_27) \Rightarrow (p V3y_27)))))) \quad (30)$$

Assume the following.

$$2.(((p \ V2Q) \Rightarrow ((p \ V0P) \Leftrightarrow (p \ V1P_27))) \wedge ((p \ V1P_27) \Rightarrow ((p \ V2Q) \Leftrightarrow (p \ V3Q_27)))) \Rightarrow ((p \ V0P) \wedge (p \ V2Q)) \Leftrightarrow ((p \ V1P_27) \wedge (p \ V3Q_27)) \quad (31)$$

Assume the following.

$$\forall A_27a.\text{nonempty } A_27a \Rightarrow (\forall V0P \in (2^{A_27a}).(\forall V1a \in A_27a.((\exists V2x \in A_27a.((V2x = V1a) \wedge (p \ (ap \ V0P \ V2x)))) \Leftrightarrow (p \ (ap \ V0P \ V1a)))))) \quad (32)$$

Assume the following.

$$\forall A_27a.\text{nonempty } A_27a \Rightarrow (\forall V0x \in (ty_2Eordinal_2Eordinal \ A_27a).(\forall V1w \in (ty_2Eordinal_2Eordinal \ A_27a).((p \ (ap \ (ap \ (c_2Ebool_2EIN \ (ty_2Eordinal_2Eordinal \ A_27a)) \ V0x) \ (ap \ (c_2Eordinal_2Epreds \ A_27a) \ V1w))) \Leftrightarrow (p \ (ap \ (ap \ (c_2Eordinal_2Eordlt \ A_27a) \ V0x) \ V1w)))))) \quad (33)$$

Assume the following.

$$\forall A_27a.\text{nonempty } A_27a \Rightarrow (\forall V0a \in (ty_2Eordinal_2Eordinal \ A_27a).(\forall V1b \in (ty_2Eordinal_2Eordinal \ A_27a).(((ap \ (c_2Eordinal_2EordSUC \ A_27a) \ V0a) = (ap \ (c_2Eordinal_2EordSUC \ A_27a) \ V1b)) \Leftrightarrow (V0a = V1b)))))) \quad (34)$$

Assume the following.

$$\forall A_27a.\text{nonempty } A_27a \Rightarrow (\forall V0P \in (2^{(ty_2Eordinal_2Eordinal \ A_27a)}).(((p \ (ap \ V0P \ (ap \ (c_2Eordinal_2EfromNat \ A_27a) \ c_2Enum_2E0))) \wedge ((\forall V1a \in (ty_2Eordinal_2Eordinal \ A_27a).((p \ (ap \ V0P \ V1a)) \Rightarrow (p \ (ap \ V0P \ (ap \ (c_2Eordinal_2EordSUC \ A_27a) \ V1a)))))) \wedge (\forall V2a \in (ty_2Eordinal_2Eordinal \ A_27a).(((ap \ (c_2Eordinal_2Eomax \ A_27a) \ (ap \ (c_2Eordinal_2Epreds \ A_27a) \ V2a)) = (c_2Eoption_2ENONE \ (ty_2Eordinal_2Eordinal \ A_27a))) \wedge ((p \ (ap \ (ap \ (c_2Eordinal_2Eordlt \ A_27a) \ (ap \ (c_2Eordinal_2EfromNat \ A_27a) \ c_2Enum_2E0)) \ V2a)) \wedge (\forall V3b \in (ty_2Eordinal_2Eordinal \ A_27a).((p \ (ap \ (ap \ (c_2Eordinal_2Eordlt \ A_27a) \ V3b) \ V2a)) \Rightarrow (p \ (ap \ V0P \ V3b)))))) \Rightarrow (p \ (ap \ V0P \ V2a)))))) \Rightarrow (\forall V4a \in (ty_2Eordinal_2Eordinal \ A_27a).(p \ (ap \ V0P \ V4a)))))) \quad (35)$$

Assume the following.

$$\forall A_27a.\text{nonempty } A_27a \Rightarrow (\forall V0a \in (ty_2Eordinal_2Eordinal \ A_27a).(((ap \ (c_2Eordinal_2Eomax \ A_27a) \ (ap \ (c_2Eordinal_2Epreds \ A_27a) \ V0a)) = (c_2Eoption_2ENONE \ (ty_2Eordinal_2Eordinal \ A_27a))) \Leftrightarrow ((ap \ (c_2Eordinal_2Esup \ A_27a) \ (ap \ (c_2Eordinal_2Epreds \ A_27a) \ V0a)) = V0a))) \quad (36)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in ty_2Enum_2Enum. (\\ & \forall V1y \in ty_2Enum_2Enum. (((ap\ (c_2Eordinal_2EfromNat\ A_27a) \\ & V0x) = (ap\ (c_2Eordinal_2EfromNat\ A_27a)\ V1y)) \Leftrightarrow (V0x = V1y)))) \end{aligned} \quad (37)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0b \in (ty_2Eordinal_2Eordinal \\ & A_27a). (((ap\ (ap\ (c_2Eordinal_2EordADD\ A_27a)\ V0b)\ (ap\ (c_2Eordinal_2EfromNat \\ & A_27a)\ c_2Enum_2E0)) = V0b) \wedge ((\forall V1a \in (ty_2Eordinal_2Eordinal \\ & A_27a). ((ap\ (ap\ (c_2Eordinal_2EordADD\ A_27a)\ V0b)\ (ap\ (c_2Eordinal_2EordSUC \\ & A_27a)\ V1a)) = (ap\ (c_2Eordinal_2EordSUC\ A_27a)\ (ap\ (ap\ (c_2Eordinal_2EordADD \\ & A_27a)\ V0b)\ V1a)))) \wedge (\forall V2a \in (ty_2Eordinal_2Eordinal\ A_27a). \\ & (((p\ (ap\ (ap\ (c_2Eordinal_2Eordlt\ A_27a)\ (ap\ (c_2Eordinal_2EfromNat \\ & A_27a)\ c_2Enum_2E0))\ V2a)) \wedge ((ap\ (c_2Eordinal_2Eomax\ A_27a)\ (\\ & ap\ (c_2Eordinal_2Epreds\ A_27a)\ V2a)) = (c_2Eoption_2ENONE\ (ty_2Eordinal_2Eordinal \\ & A_27a)))))) \Rightarrow ((ap\ (ap\ (c_2Eordinal_2EordADD\ A_27a)\ V0b)\ V2a) = (ap \\ & (c_2Eordinal_2Esup\ A_27a)\ (ap\ (ap\ (c_2Epred_set_2EIMAGE\ (ty_2Eordinal_2Eordinal \\ & A_27a)\ (ty_2Eordinal_2Eordinal\ A_27a))\ (ap\ (c_2Eordinal_2EordADD \\ & A_27a)\ V0b))\ (ap\ (c_2Eordinal_2Epreds\ A_27a)\ V2a)))))) \end{aligned} \quad (38)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0s \in (2^{A_27a}). (\forall V1t \in \\ & (2^{A_27a}). ((V0s = V1t) \Leftrightarrow (\forall V2x \in A_27a. ((p\ (ap\ (ap\ (c_2Ebool_2EIN \\ & A_27a)\ V2x)\ V0s)) \Leftrightarrow (p\ (ap\ (ap\ (c_2Ebool_2EIN\ A_27a)\ V2x)\ V1t)))))) \end{aligned} \quad (39)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow (\\ & \forall V0y \in A_27b. (\forall V1s \in (2^{A_27a}). (\forall V2f \in (A_27b^{A_27a}). \\ & ((p\ (ap\ (ap\ (c_2Ebool_2EIN\ A_27b)\ V0y)\ (ap\ (ap\ (c_2Epred_set_2EIMAGE \\ & A_27a\ A_27b)\ V2f)\ V1s))) \Leftrightarrow (\exists V3x \in A_27a. ((V0y = (ap\ V2f\ V3x)) \wedge \\ & (p\ (ap\ (ap\ (c_2Ebool_2EIN\ A_27a)\ V3x)\ V1s)))))) \end{aligned} \quad (40)$$

Theorem 1

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0a \in (ty_2Eordinal_2Eordinal \\ & A_27a). ((ap\ (ap\ (c_2Eordinal_2EordADD\ A_27a)\ (ap\ (c_2Eordinal_2EfromNat \\ & A_27a)\ c_2Enum_2E0))\ V0a) = V0a)) \end{aligned}$$