

thm_2Eordinal_2EordADD_fromNat (TMTFEB- brHHuu7x4XNoxUxnJMwRRLNB9iLH6)

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Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \tag{1}$$

Let $c_2Earithmetic_2E_2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2B \in ((ty_2Enum_2Enum)^{ty_2Enum_2Enum})^{ty_2Enum_2Enum} \tag{2}$$

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2))) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x)$

Let $c_2Enum_2EREP_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (\omega^{ty_2Enum_2Enum}) \tag{3}$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (\omega^{\omega}) \tag{4}$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{\omega}) \tag{5}$$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A.\lambda a : \iota.(\lambda V0P \in (2^{A-27a}).(ap (ap (c_2Emin_2E_3D (2^{A-27a}))) (\lambda V1f \in (2^{A-27a}).(ap V1f V0x))))$

Definition 4 We define c_2Enum_2ESUC to be $\lambda V0m \in ty_2Enum_2Enum.(ap c_2Enum_2EABS_num (c_2Enum_2EREP_num m))$

Definition 5 We define $c_2Ebool_2E_2IN$ to be $\lambda A.\lambda a : \iota.(\lambda V0x \in A.\lambda a : \iota.(\lambda V1f \in (2^{A-27a}).(ap V1f V0x))))$

Definition 6 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p \Rightarrow Q)$ of type ι .

Definition 7 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2)) (\lambda V2t \in 2.V2t))))$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (6)$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \quad (7)$$

Definition 8 We define c_2Epair_2E2C to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap\ (c_2E$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epred_set_2EGSPEC\ A_27a\ A_27b \in ((2^{A_27a})^{(ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b}}) \quad (8)$$

Definition 9 We define $c_2Epred_set_2EIMAGE$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (A_27b^{A_27a}).\lambda V1s \in ($

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \quad (9)$$

Let $ty_2Ewellorder_2Ewellorder : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Ewellorder_2Ewellorder\ A0) \quad (10)$$

Let $ty_2Eordinal_2Eordinal : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Eordinal_2Eordinal\ A0) \quad (11)$$

Let $c_2Eordinal_2Eordinal_REP_CLASS : \iota \Rightarrow \iota$ be given. Assume the following.

$$A_27a \in ((2^{(ty_2Ewellorder_2Ewellorder\ (ty_2Esum_2Esum\ ty_2Enum_2Enum\ A_27a))} (ty_2Eordinal_2Eordinal\ A_27a))) \quad (12)$$

Definition 10 We define c_2Emin_2E40 to be $\lambda A.\lambda P \in 2^A.$ if $(\exists x \in A.p\ (ap\ P\ x))$ then $(the\ (\lambda x.x \in A \wedge$

Definition 11 We define $c_2Eordinal_2Eordinal_REP$ to be $\lambda A_27a : \iota.\lambda V0a \in (ty_2Eordinal_2Eordinal\ A$

Let $c_2Ewellorder_2Ewellorder_REP : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Ewellorder_2Ewellorder_REP\ A_27a \in ((2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)} (ty_2Ewellorder_2Ewellorder\ A_27a))) \quad (13)$$

Definition 12 We define c_2Ebool_2EF to be $(ap (c_2Ebool_2E_21\ 2) (\lambda V0t \in 2.V0t))$.

Definition 13 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E))$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2ESND \\ A_27a\ A_27b \in (A_27b)^{(ty_2Epair_2Eprod\ A_27a\ A_27b)} \end{aligned} \quad (14)$$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EFST \\ A_27a\ A_27b \in (A_27a)^{(ty_2Epair_2Eprod\ A_27a\ A_27b)} \end{aligned} \quad (15)$$

Definition 14 We define $c_2Epair_2EUNCURRY$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0f \in ((A_27c)^{A_27a})$

Definition 15 We define $c_2Eset_relation_2Estrict$ to be $\lambda A_27a : \iota.\lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})$

Definition 16 We define $c_2Ewellorder_2Eiseg$ to be $\lambda A_27a : \iota.\lambda V0w \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 17 We define $c_2Eset_relation_2Errestrict$ to be $\lambda A_27a : \iota.\lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})$

Let $c_2Ewellorder_2Ewellorder_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow c_2Ewellorder_2Ewellorder_ABS \\ A_27a \in ((ty_2Ewellorder_2Ewellorder\ A_27a)^{(2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})}) \end{aligned} \quad (16)$$

Definition 18 We define $c_2Ewellorder_2Ewobound$ to be $\lambda A_27a : \iota.\lambda V0x \in A_27a.\lambda V1w \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 19 We define $c_2Ebool_2E_3F$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap V0P (ap (c_2Emin_2E_40$

Definition 20 We define $c_2Eset_relation_2Erange$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27b)})$

Definition 21 We define $c_2Eset_relation_2Edomain$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27b)})$

Definition 22 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21\ 2) (\lambda V2t \in 2.V2t$

Definition 23 We define $c_2Epred_set_2EUNION$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap (c_2Emin_2E_40$

Definition 24 We define $c_2Ewellorder_2EelsOf$ to be $\lambda A_27a : \iota.\lambda V0w \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 25 We define $c_2Ewellorder_2Eorderiso$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0w1 \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 26 We define $c_2Ewellorder_2Eorderlt$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0w1 \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 27 We define $c_2Eordinal_2Eordlt$ to be $\lambda A_27a : \iota.\lambda V0T1 \in (ty_2Eordinal_2Eordinal\ A_27a).$

Definition 28 We define $c_2Eordinal_2Epreds$ to be $\lambda A_27a : \iota.\lambda V0w \in (ty_2Eordinal_2Eordinal\ A_27a).$

Definition 29 We define $c_2Epred_set_2EBIGUNION$ to be $\lambda A_27a : \iota.\lambda V0P \in (2^{(2^{A_27a})}).(ap (c_2Epred_set_2EBIGUNION A_27a) P)$

Definition 30 We define $c_2Eordinal_2Eoleast$ to be $\lambda A_27a : \iota.\lambda V0P \in (2^{(ty_2Eordinal_2Eordinal A_27a)}).$

Definition 31 We define $c_2Eordinal_2Esup$ to be $\lambda A_27a : \iota.\lambda V0ordset \in (2^{(ty_2Eordinal_2Eordinal A_27a)}).$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \quad (17)$$

Definition 32 We define c_2Eone_2Eone to be $(ap (c_2Emin_2E40 ty_2Eone_2Eone) (\lambda V0x \in ty_2Eone_2Eone))$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum \\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \end{aligned} \quad (18)$$

Definition 33 We define c_2Esum_2EINR to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27b.(ap (c_2Esum_2EABS_sum A_27a A_27b) e)$

Let $ty_2Eoption_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Eoption_2Eoption\ A0) \quad (19)$$

Let $c_2Eoption_2Eoption_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} \forall A_27a.nonempty\ A_27a \Rightarrow c_2Eoption_2Eoption_ABS\ A_27a \in \\ ((ty_2Eoption_2Eoption\ A_27a)^{(ty_2Esum_2Esum\ A_27a\ ty_2Eone_2Eone)}) \end{aligned} \quad (20)$$

Definition 34 We define $c_2Eoption_2ENONE$ to be $\lambda A_27a : \iota.(ap (c_2Eoption_2Eoption_ABS\ A_27a) (ty_2Eone_2Eone))$

Definition 35 We define $c_2Epred_set_2EEMPTY$ to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.c_2Ebool_2Ebool_2Efalse)$

Definition 36 We define $c_2Epred_set_2EINSERT$ to be $\lambda A_27a : \iota.\lambda V0x \in A_27a.\lambda V1s \in (2^{A_27a}).(ap (c_2Epred_set_2EINSERT A_27a x) s)$

Definition 37 We define $c_2Eset_relation_2Emaximal_elements$ to be $\lambda A_27a : \iota.\lambda V0xs \in (2^{A_27a}).\lambda V1r \in (2^{A_27a}).$

Definition 38 We define c_2Esum_2EINL to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27a.(ap (c_2Esum_2EABS_sum A_27a A_27b) e)$

Definition 39 We define $c_2Eoption_2ESOME$ to be $\lambda A_27a : \iota.\lambda V0x \in A_27a.(ap (c_2Eoption_2Eoption_ABS\ A_27a) x)$

Definition 40 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.))$

Definition 41 We define $c_2Eoption_2ESome$ to be $\lambda A_27a : \iota.\lambda V0P \in (2^{A_27a}).(ap (ap (ap (c_2Ebool_2ECOND A_27a) P) (ty_2Eone_2Eone)) (ty_2Eone_2Eone))$

Definition 42 We define $c_2Eordinal_2Eomax$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{(ty_2Eordinal_2Eordinal A_27a)}).$

Definition 43 We define $c_2Eordinal_2EordSUC$ to be $\lambda A_27a : \iota.\lambda V0a \in (ty_2Eordinal_2Eordinal A_27a).$

Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in \omega \quad (21)$$

Definition 44 We define c_Enum_2E0 to be $(ap\ c_Enum_2EABS_num\ c_Enum_2EZERO_REP)$.

Let $c_Eordinal_2EfromNat : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_Eordinal_2EfromNat\ A_27a \in ((ty_2Eordinal_2Eordinal\ A_27a)^{ty_2Enum_2Enum}) \quad (22)$$

Let $c_Eordinal_2EordADD : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_Eordinal_2EordADD\ A_27a \in (((ty_2Eordinal_2Eordinal\ A_27a)^{(ty_2Eordinal_2Eordinal\ A_27a)})^{(ty_2Eordinal_2Eordinal\ A_27a)}) \quad (23)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\\ & \quad ((ap\ (ap\ c_2Earithmetic_2E_2B\ c_2Enum_2E0)\ V0m) = V0m) \wedge (((ap\ (\\ & \quad ap\ c_2Earithmetic_2E_2B\ V0m)\ c_2Enum_2E0) = V0m) \wedge (((ap\ (ap\ c_2Earithmetic_2E_2B \\ & \quad (ap\ c_2Enum_2ESUC\ V0m))\ V1n) = (ap\ c_2Enum_2ESUC\ (ap\ (ap\ c_2Earithmetic_2E_2B \\ & \quad V0m)\ V1n))) \wedge ((ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ (ap\ c_2Enum_2ESUC \\ & \quad V1n)) = (ap\ c_2Enum_2ESUC\ (ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ V1n))))))))) \end{aligned} \quad (24)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\\ & \quad (ap\ (ap\ c_2Earithmetic_2E_2B\ V0m)\ V1n) = (ap\ (ap\ c_2Earithmetic_2E_2B \\ & \quad V1n)\ V0m)))) \end{aligned} \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0m \in ty_2Enum_2Enum. (\forall V1n \in ty_2Enum_2Enum. (\\ & \quad \forall V2p \in ty_2Enum_2Enum. (((ap\ (ap\ c_2Earithmetic_2E_2B\ V0m) \\ & \quad V2p) = (ap\ (ap\ c_2Earithmetic_2E_2B\ V1n)\ V2p)) \Leftrightarrow (V0m = V1n)))))) \end{aligned} \quad (26)$$

Assume the following.

$$True \quad (27)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (28)$$

Assume the following.

$$\begin{aligned} & (\forall V0P \in (2^{ty_2Enum_2Enum}). (((p\ (ap\ V0P\ c_2Enum_2E0)) \wedge \\ & \quad (\forall V1n \in ty_2Enum_2Enum. ((p\ (ap\ V0P\ V1n)) \Rightarrow (p\ (ap\ V0P\ (ap\ c_2Enum_2ESUC \\ & \quad V1n)))))) \Rightarrow (\forall V2n \in ty_2Enum_2Enum. (p\ (ap\ V0P\ V2n)))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0n \in ty_2Enum_2Enum. (\\ (ap\ (c_2Eordinal_2EfromNat\ A.27a)\ (ap\ c_2Enum_2ESUC\ V0n)) = (ap \\ (c_2Eordinal_2EordSUC\ A.27a)\ (ap\ (c_2Eordinal_2EfromNat\ A.27a) \\ V0n)))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0a \in (ty_2Eordinal_2Eordinal \\ A.27a). (\forall V1b \in (ty_2Eordinal_2Eordinal\ A.27a). (((ap\ (\\ c_2Eordinal_2EordSUC\ A.27a)\ V0a) = (ap\ (c_2Eordinal_2EordSUC \\ A.27a)\ V1b)) \Leftrightarrow (V0a = V1b)))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0x \in ty_2Enum_2Enum. (\\ \forall V1y \in ty_2Enum_2Enum. (((ap\ (c_2Eordinal_2EfromNat\ A.27a) \\ V0x) = (ap\ (c_2Eordinal_2EfromNat\ A.27a)\ V1y)) \Leftrightarrow (V0x = V1y)))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0b \in (ty_2Eordinal_2Eordinal \\ A.27a). (((ap\ (ap\ (c_2Eordinal_2EordADD\ A.27a)\ V0b)\ (ap\ (c_2Eordinal_2EfromNat \\ A.27a)\ c_2Enum_2E0)) = V0b) \wedge ((\forall V1a \in (ty_2Eordinal_2Eordinal \\ A.27a). ((ap\ (ap\ (c_2Eordinal_2EordADD\ A.27a)\ V0b)\ (ap\ (c_2Eordinal_2EordSUC \\ A.27a)\ V1a)) = (ap\ (c_2Eordinal_2EordSUC\ A.27a)\ (ap\ (ap\ (c_2Eordinal_2EordADD \\ A.27a)\ V0b)\ V1a)))) \wedge (\forall V2a \in (ty_2Eordinal_2Eordinal\ A.27a). \\ (((p\ (ap\ (ap\ (c_2Eordinal_2Eordlt\ A.27a)\ (ap\ (c_2Eordinal_2EfromNat \\ A.27a)\ c_2Enum_2E0))\ V2a)) \wedge ((ap\ (c_2Eordinal_2Eomax\ A.27a)\ (\\ ap\ (c_2Eordinal_2Epreds\ A.27a)\ V2a)) = (c_2Eoption_2ENONE\ (ty_2Eordinal_2Eordinal \\ A.27a)))))) \Rightarrow ((ap\ (ap\ (c_2Eordinal_2EordADD\ A.27a)\ V0b)\ V2a) = (ap \\ (c_2Eordinal_2Esup\ A.27a)\ (ap\ (ap\ (c_2Epred_set_2EIMAGE\ (ty_2Eordinal_2Eordinal \\ A.27a)\ (ty_2Eordinal_2Eordinal\ A.27a))\ (ap\ (c_2Eordinal_2EordADD \\ A.27a)\ V0b))\ (ap\ (c_2Eordinal_2Epreds\ A.27a)\ V2a)))))) \end{aligned} \quad (33)$$

Theorem 1

$$\begin{aligned} \forall A.27a.nonempty\ A.27a \Rightarrow (\forall V0n \in ty_2Enum_2Enum. (\\ \forall V1m \in ty_2Enum_2Enum. ((ap\ (ap\ (c_2Eordinal_2EordADD\ A.27a) \\ (ap\ (c_2Eordinal_2EfromNat\ A.27a)\ V0n))\ (ap\ (c_2Eordinal_2EfromNat \\ A.27a)\ V1m)) = (ap\ (c_2Eordinal_2EfromNat\ A.27a)\ (ap\ (ap\ c_2Earithmic_2E_2B \\ V0n)\ V1m)))) \end{aligned}$$