

thm_2Eordinal_2EordMULT__0R
(TMXV3htqbLUSwKJaDDgaQgu35zHsQsUfc4k)

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Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_2ET$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \tag{1}$$

Definition 3 We define $c_2Ebool_2E_2EIN$ to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap\ V1f\ V0x)))$

Definition 4 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p\ P \Rightarrow p\ Q)$ of type ι .

Definition 5 We define $c_2Ebool_2E_2E21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a})))$

Definition 6 We define $c_2Ebool_2E_2E2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_2E21\ 2)) (\lambda V2t \in 2.V2t)))$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \tag{2}$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \tag{3}$$

Definition 7 We define $c_2Epair_2E_2EC$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap (c_2Emin_2E_3D (2^{A_27a})) (\lambda V2z \in 2.V2z)))$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epred_set_2EGSPEC\ A_27a\ A_27b \in ((2^{A_27a})^{(ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b}}) \tag{4}$$

Definition 8 We define $c_2Epred_set_2EIMAGE$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (A_27b^{A_27a}).\lambda V1s \in ($

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \quad (5)$$

Let $ty_2Ewellorder_2Ewellorder : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Ewellorder_2Ewellorder\ A0) \quad (6)$$

Let $ty_2Eordinal_2Eordinal : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Eordinal_2Eordinal\ A0) \quad (7)$$

Let $c_2Eordinal_2Eordinal_REP_CLASS : \iota \Rightarrow \iota$ be given. Assume the following.

$$A_27a \in ((2^{(ty_2Ewellorder_2Ewellorder\ (ty_2Esum_2Esum\ ty_2Enum_2Enum\ A_27a))})^{(ty_2Eordinal_2Eordinal\ A_27a)}) \quad (8)$$

Definition 9 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.$ if $(\exists x \in A.p\ (ap\ P\ x))$ then (the $(\lambda x.x \in A \wedge p$ of type $\iota \Rightarrow \iota$).

Definition 10 We define $c_2Eordinal_2Eordinal_REP$ to be $\lambda A_27a : \iota.\lambda V0a \in (ty_2Eordinal_2Eordinal\ A$

Let $c_2Ewellorder_2Ewellorder_REP : \iota \Rightarrow \iota$ be given. Assume the following.

$$A_27a \in ((2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})^{(ty_2Ewellorder_2Ewellorder\ A_27a)}) \quad (9)$$

Definition 11 We define c_2Ebool_2EF to be $(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 12 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap\ (ap\ c_2Emin_2E_3D_3D_3E\ V0t)\ c_2Ebool_2E$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2ESND\ A_27a\ A_27b \in (A_27b^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (10)$$

Let $c_2Epair_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EFST\ A_27a\ A_27b \in (A_27a^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \quad (11)$$

Definition 13 We define $c_2Epair_2EUNCURRY$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.\lambda V0f \in ((A_27c^{A_27$

Definition 14 We define $c_Eset_relation_Estrict$ to be $\lambda A_27a : \iota. \lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})$

Definition 15 We define $c_Ewellorder_Eiseg$ to be $\lambda A_27a : \iota. \lambda V0w \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 16 We define $c_Eset_relation_Erestrict$ to be $\lambda A_27a : \iota. \lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})$

Let $c_2Ewellorder_2Ewellorder_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} & \forall A_27a. nonempty\ A_27a \Rightarrow c_2Ewellorder_2Ewellorder_ABS \\ & A_27a \in ((ty_2Ewellorder_2Ewellorder\ A_27a)^{(2^{(ty_2Epair_2Eprod\ A_27a\ A_27a)})}) \end{aligned} \quad (12)$$

Definition 17 We define $c_2Ewellorder_2Ewobound$ to be $\lambda A_27a : \iota. \lambda V0x \in A_27a. \lambda V1w \in (ty_2Ewellorder\ A_27a)$

Definition 18 We define $c_2Ebool_2E_3F$ to be $\lambda A_27a : \iota. (\lambda V0P \in (2^{A_27a}). (ap\ V0P\ (ap\ (c_2Emin_2E_40\ A_27a\ P))))$

Definition 19 We define $c_Eset_relation_Erange$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27b)})$

Definition 20 We define $c_Eset_relation_Edomain$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0r \in (2^{(ty_2Epair_2Eprod\ A_27a\ A_27b)})$

Definition 21 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c_2Ebool_2E_21\ 2)\ t1\ t2)))$

Definition 22 We define $c_2Epred_set_2EUNION$ to be $\lambda A_27a : \iota. \lambda V0s \in (2^{A_27a}). \lambda V1t \in (2^{A_27a}). (ap\ (c_2Epred_set_2EUNION\ s\ t))$

Definition 23 We define $c_2Ewellorder_2EelsOf$ to be $\lambda A_27a : \iota. \lambda V0w \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 24 We define $c_2Ewellorder_2Eorderiso$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0w1 \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 25 We define $c_2Ewellorder_2Eorderlt$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0w1 \in (ty_2Ewellorder_2Ewellorder\ A_27a)$

Definition 26 We define $c_2Eordinal_2Eordlt$ to be $\lambda A_27a : \iota. \lambda V0T1 \in (ty_2Eordinal_2Eordinal\ A_27a)$

Definition 27 We define $c_2Eordinal_2Epreds$ to be $\lambda A_27a : \iota. \lambda V0w \in (ty_2Eordinal_2Eordinal\ A_27a)$

Definition 28 We define $c_2Epred_set_2EBIGUNION$ to be $\lambda A_27a : \iota. \lambda V0P \in (2^{(2^{A_27a})}). (ap\ (c_2Epred_set_2EBIGUNION\ P))$

Definition 29 We define $c_2Eordinal_2Eoleast$ to be $\lambda A_27a : \iota. \lambda V0P \in (2^{(ty_2Eordinal_2Eordinal\ A_27a)})$

Definition 30 We define $c_2Eordinal_2Esup$ to be $\lambda A_27a : \iota. \lambda V0ordset \in (2^{(ty_2Eordinal_2Eordinal\ A_27a)})$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \quad (13)$$

Definition 31 We define c_2Eone_2Eone to be $(ap\ (c_2Emin_2E_40\ ty_2Eone_2Eone)\ (\lambda V0x \in ty_2Eone_2Eone))$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} & \forall A_27a. nonempty\ A_27a \Rightarrow \forall A_27b. nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum \\ & A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \end{aligned} \quad (14)$$

Definition 32 We define c_Esum_2EINR to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0e \in A_27b. (ap (c_Esum_2EABS$
Let $ty_2Eoption_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow nonempty (ty_2Eoption_2Eoption A0) \quad (15)$$

Let $c_2Eoption_2Eoption_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a. nonempty A_27a \Rightarrow c_2Eoption_2Eoption_ABS A_27a \in ((ty_2Eoption_2Eoption A_27a)^{(ty_2Esum_2Esum A_27a ty_2Eone_2Eone)}) \quad (16)$$

Definition 33 We define $c_2Eoption_2ENONE$ to be $\lambda A_27a : \iota. (ap (c_2Eoption_2Eoption_ABS A_27a) (c_2Eone_2Eone))$

Definition 34 We define $c_2Epred_set_2EEMPTY$ to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a. c_2Ebool_2EF)$.

Definition 35 We define $c_2Epred_set_2EINSERT$ to be $\lambda A_27a : \iota. \lambda V0x \in A_27a. \lambda V1s \in (2^{A_27a}). (ap (c_2Ebool_2EF) (c_2Ebool_2EF))$

Definition 36 We define $c_2Eset_relation_2Emaximal_elements$ to be $\lambda A_27a : \iota. \lambda V0xs \in (2^{A_27a}). \lambda V1r \in (2^{A_27a}). (c_2Ebool_2EF)$

Definition 37 We define c_Esum_2EINL to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0e \in A_27a. (ap (c_Esum_2EABS$

Definition 38 We define $c_2Eoption_2ESOME$ to be $\lambda A_27a : \iota. \lambda V0x \in A_27a. (ap (c_2Eoption_2Eoption_ABS A_27a) (c_2Ebool_2EF))$

Definition 39 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A_27a. (\lambda V2t2 \in A_27a. (c_2Ebool_2EF))$

Definition 40 We define $c_2Eoption_2ESOME$ to be $\lambda A_27a : \iota. \lambda V0P \in (2^{A_27a}). (ap (ap (ap (c_2Ebool_2EF) (c_2Ebool_2EF)) (c_2Ebool_2EF)) (c_2Ebool_2EF))$

Definition 41 We define $c_2Eordinal_2Eomax$ to be $\lambda A_27a : \iota. \lambda V0s \in (2^{(ty_2Eordinal_2Eordinal A_27a)}). (ap (c_2Ebool_2EF) (c_2Ebool_2EF))$

Let $c_2Eordinal_2EordADD : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a. nonempty A_27a \Rightarrow c_2Eordinal_2EordADD A_27a \in ((ty_2Eordinal_2Eordinal A_27a)^{(ty_2Eordinal_2Eordinal A_27a)}) \quad (17)$$

Definition 42 We define $c_2Eordinal_2EordSUC$ to be $\lambda A_27a : \iota. \lambda V0a \in (ty_2Eordinal_2Eordinal A_27a). (c_2Ebool_2EF)$

Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in \omega \quad (18)$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{\omega}) \quad (19)$$

Definition 43 We define c_2Enum_2E0 to be $(ap c_2Enum_2EABS_num c_2Enum_2EZERO_REP)$.

Let $c_2Eordinal_2EfromNat : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Eordinal_2EfromNat\ A_27a \in ((ty_2Eordinal_2Eordinal\ A_27a)^{ty_2Enum_2Enum}) \quad (20)$$

Let $c_2Eordinal_2EordMULT : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Eordinal_2EordMULT\ A_27a \in (((ty_2Eordinal_2Eordinal\ A_27a)^{(ty_2Eordinal_2Eordinal\ A_27a)})^{(ty_2Eordinal_2Eordinal\ A_27a)}) \quad (21)$$

Assume the following.

$$True \quad (22)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in 2. ((\forall V1x \in A_27a.(p\ V0t) \Leftrightarrow (p\ V0t)))) \quad (23)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (24)$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in ty_2Enum_2Enum. (\forall V1y \in ty_2Enum_2Enum. (((ap\ (c_2Eordinal_2EfromNat\ A_27a)\ V0x) = (ap\ (c_2Eordinal_2EfromNat\ A_27a)\ V1y)) \Leftrightarrow (V0x = V1y)))) \quad (25)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0b \in (ty_2Eordinal_2Eordinal\ A_27a). (((ap\ (ap\ (c_2Eordinal_2EordMULT\ A_27a)\ V0b)\ (ap\ (c_2Eordinal_2EfromNat\ A_27a)\ c_2Enum_2E0)) = (ap\ (c_2Eordinal_2EfromNat\ A_27a)\ c_2Enum_2E0)) \wedge \\ & ((\forall V1a \in (ty_2Eordinal_2Eordinal\ A_27a). ((ap\ (ap\ (c_2Eordinal_2EordMULT\ A_27a)\ V0b)\ (ap\ (c_2Eordinal_2EordSUC\ A_27a)\ V1a)) = (ap\ (ap\ (c_2Eordinal_2EordADD\ A_27a)\ (ap\ (ap\ (c_2Eordinal_2EordMULT\ A_27a)\ V0b)\ V1a))\ V0b))) \wedge \\ & (\forall V2a \in (ty_2Eordinal_2Eordinal\ A_27a). (((p\ (ap\ (ap\ (c_2Eordinal_2Eordlt\ A_27a)\ (ap\ (c_2Eordinal_2EfromNat\ A_27a)\ c_2Enum_2E0))\ V2a)) \wedge \\ & ((ap\ (c_2Eordinal_2Eomax\ A_27a)\ (ap\ (c_2Eordinal_2Epreds\ A_27a)\ V2a)) = (c_2Eoption_2ENONE\ (ty_2Eordinal_2Eordinal\ A_27a)))) \Rightarrow \\ & ((ap\ (ap\ (c_2Eordinal_2EordMULT\ A_27a)\ V0b)\ V2a) = (ap\ (c_2Eordinal_2Esup\ A_27a)\ (ap\ (ap\ (c_2Epred_set_2EIMAGE\ (ty_2Eordinal_2Eordinal\ A_27a)\ (ty_2Eordinal_2Eordinal\ A_27a))\ (ap\ (c_2Eordinal_2EordMULT\ A_27a)\ V0b))\ (ap\ (c_2Eordinal_2Epreds\ A_27a)\ V2a)))))) \end{aligned} \quad (26)$$

Theorem 1

$$\forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0a \in (ty_2Eordinal_2Eordinal\ A_27a). ((ap\ (ap\ (c_2Eordinal_2EordMULT\ A_27a)\ V0a)\ (ap\ (c_2Eordinal_2EfromNat\ A_27a)\ c_2Enum_2E0)) = (ap\ (c_2Eordinal_2EfromNat\ A_27a)\ c_2Enum_2E0)))$$