

thm_2Epath_2EfirstP__at__zero
(TMaFpn5q8BRaS8EEPYNKdsmb3sSUhrccmP5)

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Definition 1 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 2 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 3 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 4 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 5 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \tag{1}$$

Let $ty_2Ellist_2Ellist : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty_2Ellist_2Ellist A0) \tag{2}$$

Let $ty_2Epath_2Epath : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epath_2Epath A0 A1) \tag{3}$$

Let $c_2Epath_2EfromPath : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epath_2EfromPath A_27a A_27b \in ((ty_2Epair_2Eprod A_27a (ty_2Ellist_2Ellist (ty_2Epair_2Eprod A_27b A_27a)))^{(ty_2Epath_2Epath A_27a A_27b)}) \tag{4}$$

Let $c_2Epair_2EFSST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epair_2EFSST A_27a A_27b \in (A_27a^{(ty_2Epair_2Eprod A_27a A_27b)}) \tag{5}$$

Definition 6 We define $c_2Epath_2Efirst$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0p \in (ty_2Epath_2Epath\ A_27a\ A_27b)$.
Let $c_2Enum_2EZERO_REP : \iota$ be given. Assume the following.

$$c_2Enum_2EZERO_REP \in \omega \tag{6}$$

Let $ty_2Enum_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Enum_2Enum \tag{7}$$

Let $c_2Enum_2EABS_num : \iota$ be given. Assume the following.

$$c_2Enum_2EABS_num \in (ty_2Enum_2Enum^{\omega}) \tag{8}$$

Definition 7 We define c_2Enum_2E0 to be $(ap\ c_2Enum_2EABS_num\ c_2Enum_2EZERO_REP)$.

Definition 8 We define $c_2Earithmetic_2EZERO$ to be c_2Enum_2E0 .

Let $c_2Enum_2EREP_num : \iota$ be given. Assume the following.

$$c_2Enum_2EREP_num \in (\omega^{ty_2Enum_2Enum}) \tag{9}$$

Let $c_2Enum_2ESUC_REP : \iota$ be given. Assume the following.

$$c_2Enum_2ESUC_REP \in (\omega^{\omega}) \tag{10}$$

Definition 9 We define c_2Enum_2ESUC to be $\lambda V0m \in ty_2Enum_2Enum. (ap\ c_2Enum_2EABS_num\ m)$

Let $c_2Earithmetic_2E_2B : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2B \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \tag{11}$$

Definition 10 We define $c_2Earithmetic_2EBIT1$ to be $\lambda V0n \in ty_2Enum_2Enum. (ap\ (ap\ c_2Earithmetic_2E_2B\ n))$

Definition 11 We define $c_2Earithmetic_2ENUMERAL$ to be $\lambda V0x \in ty_2Enum_2Enum. V0x$.

Let $c_2Earithmetic_2E_2D : \iota$ be given. Assume the following.

$$c_2Earithmetic_2E_2D \in ((ty_2Enum_2Enum^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}) \tag{12}$$

Definition 12 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c_2Ebool_2E_21\ 2)\ t2)))$

Let $c_2Epair_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\begin{aligned} & \forall A_27a. nonempty\ A_27a \Rightarrow \forall A_27b. nonempty\ A_27b \Rightarrow c_2Epair_2ESND \\ & A_27a\ A_27b \in (A_27b^{(ty_2Epair_2Eprod\ A_27a\ A_27b)}) \end{aligned} \tag{13}$$

Definition 13 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c_2Ebool_2E_21\ 2)\ t2)))$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \quad (14)$$

Definition 14 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0x \in A_27a. \lambda V1y \in A_27b. (ap\ (c_2Epair_2EABS_prod\ A_27a\ A_27b)\ x\ y)$

Let $ty_2Eoption_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Eoption_2Eoption\ A0) \quad (15)$$

Let $c_2Ellist_2Ellist_rep : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Ellist_2Ellist_rep\ A_27a \in ((ty_2Eoption_2Eoption\ A_27a)^{ty_2Enum_2Enum})^{(ty_2Ellist_2Ellist\ A_27a)} \quad (16)$$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \quad (17)$$

Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Esum_2Esum\ A0\ A1) \quad (18)$$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Esum_2EABS_sum\ A_27a\ A_27b \in ((ty_2Esum_2Esum\ A_27a\ A_27b)^{((2^{A_27b})^{A_27a})^2}) \quad (19)$$

Definition 15 We define c_2Esum_2EINL to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0e \in A_27a. (ap\ (c_2Esum_2EABS_sum\ A_27a\ A_27b)\ e)$

Let $c_2Eoption_2Eoption_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Eoption_2Eoption_ABS\ A_27a \in ((ty_2Eoption_2Eoption\ A_27a)^{ty_2Esum_2Esum\ A_27a\ ty_2Eone_2Eone}) \quad (20)$$

Definition 16 We define $c_2Eoption_2ESOME$ to be $\lambda A_27a : \iota. \lambda V0x \in A_27a. (ap\ (c_2Eoption_2Eoption_ABS\ A_27a)\ x)$

Definition 17 We define $c_2Emin_2E_40$ to be $\lambda A. \lambda P \in 2^A. \text{if } (\exists x \in A. P\ x) \text{ then } (the\ (\lambda x. x \in A \wedge P\ x)) \text{ of type } \iota \Rightarrow \iota.$

Definition 18 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A_27a. (\lambda V2t2 \in A_27a. (and\ t1\ t2))))$

Let $c_2Ellist_2Ellist_abs : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Ellist_2Ellist_abs\ A_27a \in ((ty_2Ellist_2Ellist\ A_27a)^{(ty_2Eoption_2Eoption\ A_27a)^{ty_2Enum_2Enum}}) \quad (21)$$

Definition 19 We define $c_2Ellist_2ELCONS$ to be $\lambda A_27a : \iota.\lambda V0h \in A_27a.\lambda V1t \in (ty_2Ellist_2Ellist A_27a)$.
Let $c_2Epath_2EtoPath : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epath_2EtoPath A_27a A_27b \in ((ty_2Epath_2Epath A_27a A_27b)^{(ty_2Epair_2Eprod A_27a (ty_2Ellist_2Ellist (ty_2Epair_2Eprod A_27a A_27b))})) \quad (22)$$

Definition 20 We define c_2Epath_2Epcns to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1r \in A_27b.\lambda V2p \in A_27b$.

Definition 21 We define c_2Eone_2Eone to be $(ap (c_2Emin_2E_40 ty_2Eone_2Eone) (\lambda V0x \in ty_2Eone_2Eone))$.

Definition 22 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_7E))$.

Definition 23 We define c_2Esum_2EINR to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27b.(ap (c_2Esum_2EABS A_27a A_27b))$.

Definition 24 We define $c_2Eoption_2ENONE$ to be $\lambda A_27a : \iota.(ap (c_2Eoption_2Eoption_ABS A_27a) (\lambda V0n \in ty_2Eoption_2ENONE))$.

Definition 25 We define $c_2Ellist_2ELNIL$ to be $\lambda A_27a : \iota.(ap (c_2Ellist_2Ellist_abs A_27a) (\lambda V0n \in ty_2Ellist_2ELNIL))$.

Definition 26 We define $c_2Epath_2Estopped_at$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.(ap (c_2Epath_2Estopped_at A_27a A_27b))$.

Let $c_2Epath_2Eel : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epath_2Eel A_27a A_27b \in ((A_27a)^{(ty_2Epath_2Epath A_27a A_27b)})^{ty_2Enum_2Enum} \quad (23)$$

Definition 27 We define $c_2Ebool_2E_3F$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap V0P (ap (c_2Emin_2E_40 A_27a))))$.

Definition 28 We define $c_2Eprim_rec_2E_3C$ to be $\lambda V0m \in ty_2Enum_2Enum.\lambda V1n \in ty_2Enum_2Enum$.

Let $ty_2Elist_2Elist : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty_2Elist_2Elist A0) \quad (24)$$

Definition 29 We define $c_2Ellist_2Ellength_rel$ to be $\lambda A_27a : \iota.(\lambda V0a0 \in (ty_2Ellist_2Ellist A_27a)).(\lambda V0a1 \in (ty_2Ellist_2Ellist A_27a))$.

Definition 30 We define $c_2Ellist_2ELFINITE$ to be $\lambda A_27a : \iota.(\lambda V0a0 \in (ty_2Ellist_2Ellist A_27a)).(ap (c_2Ellist_2ELFINITE A_27a) V0a0)$.

Definition 31 We define $c_2Ellist_2ELLENGTH$ to be $\lambda A_27a : \iota.\lambda V0ll \in (ty_2Ellist_2Ellist A_27a).(ap (c_2Ellist_2ELLENGTH A_27a) V0ll)$.

Let $c_2Eoption_2ETHE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Eoption_2ETHE A_27a \in (A_27a)^{(ty_2Eoption_2Eoption A_27a)} \quad (25)$$

Let $c_2Ellist_2ELTAKE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Ellist_2ELTAKE A_27a \in (((ty_2Eoption_2Eoption A_27a)^{(ty_2Elist_2Elist A_27a)})^{(ty_2Ellist_2Ellist A_27a)})^{ty_2Enum_2Enum} \quad (26)$$

Definition 32 We define $c_2Elist_2EtoList$ to be $\lambda A_27a : \iota.\lambda V0ll \in (ty_2Ellist_2Ellist A_27a).(ap (ap (ap$
Let $c_2Elist_2ELENGTH : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Elist_2ELENGTH A_27a \in (ty_2Enum_2Enum^{(ty_2Elist_2Elist A_27a)}) \quad (27)$$

Definition 33 We define $c_2Epath_2Efinite$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0sigma \in (ty_2Epath_2Epath A_27a$

Definition 34 We define $c_2Epath_2Elength$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0p \in (ty_2Epath_2Epath A_27a$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epred_set_2EGSPEC A_27a A_27b \in ((2^{A_27a})^{((ty_2Epair_2Eprod A_27a 2)^{A_27b})}) \quad (28)$$

Definition 35 We define c_2Epath_2EPL to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0p \in (ty_2Epath_2Epath A_27a A$

Definition 36 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota.(\lambda V0x \in A_27a.(\lambda V1f \in (2^{A_27a}).(ap V1f V0x))$

Definition 37 We define $c_2Epath_2EfirstP_at$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0P \in (2^{A_27a}).\lambda V1p \in (ty_2E$

Assume the following.

$$(\forall V0m \in ty_2Enum_2Enum.(((ap (ap c_2Earithmetic_2E_2D c_2Enum_2E0) V0m) = c_2Enum_2E0) \wedge ((ap (ap c_2Earithmetic_2E_2D V0m) c_2Enum_2E0) = V0m))) \quad (29)$$

Assume the following.

$$True \quad (30)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (31)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (32)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A_27a.(p V0t) \Leftrightarrow (p V0t))) \quad (33)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (34)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \vee (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \vee True) \Leftrightarrow True) \wedge \\
& (((False \vee (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee False) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee \\
& (p \ V0t)) \Leftrightarrow (p \ V0t))))))
\end{aligned} \tag{35}$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a.((V0x = V0x) \Leftrightarrow True)) \tag{36}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty \ A_27a \Rightarrow \forall A_27b.nonempty \ A_27b \Rightarrow (\\
& \forall V0P \in (2^{(ty_2Epath_2Epath \ A_27a \ A_27b)}).((\forall V1p \in \\
& (ty_2Epath_2Epath \ A_27a \ A_27b).(p \ (ap \ V0P \ V1p))) \Leftrightarrow ((\forall V2x \in \\
& A_27a.(p \ (ap \ V0P \ (ap \ (c_2Epath_2Estopped_at \ A_27a \ A_27b) \ V2x)))) \wedge \\
& (\forall V3x \in A_27a.(\forall V4r \in A_27b.(\forall V5p \in (ty_2Epath_2Epath \\
& A_27a \ A_27b).(p \ (ap \ V0P \ (ap \ (ap \ (ap \ (c_2Epath_2Epcns \ A_27a \ A_27b) \\
& V3x) \ V4r) \ V5p))))))))))
\end{aligned} \tag{37}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty \ A_27a \Rightarrow \forall A_27b.nonempty \ A_27b \Rightarrow (\\
& (\forall V0x \in A_27a.((ap \ (c_2Epath_2Efirst \ A_27a \ A_27b) \ (ap \ (c_2Epath_2Estopped_at \\
& A_27a \ A_27b) \ V0x)) = V0x)) \wedge (\forall V1x \in A_27a.(\forall V2r \in A_27b. \\
& (\forall V3p \in (ty_2Epath_2Epath \ A_27a \ A_27b).((ap \ (c_2Epath_2Efirst \\
& A_27a \ A_27b) \ (ap \ (ap \ (ap \ (c_2Epath_2Epcns \ A_27a \ A_27b) \ V1x) \ V2r) \\
& V3p)) = V1x))))))
\end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty \ A_27a \Rightarrow \forall A_27b.nonempty \ A_27b \Rightarrow (\\
& (\forall V0P \in (2^{A_27a}).(\forall V1x \in A_27a.(\forall V2n \in ty_2Enum_2Enum. \\
& ((p \ (ap \ (ap \ (ap \ (c_2Epath_2EfirstP_at \ A_27a \ A_27b) \ V0P) \ (ap \ (c_2Epath_2Estopped_at \\
& A_27a \ A_27b) \ V1x)) \ V2n)) \Leftrightarrow ((V2n = c_2Enum_2E0) \wedge (p \ (ap \ V0P \ V1x)))))) \wedge \\
& (\forall V3P \in (2^{A_27a}).(\forall V4n \in ty_2Enum_2Enum.(\forall V5x \in \\
& A_27a.(\forall V6r \in A_27b.(\forall V7p \in (ty_2Epath_2Epath \ A_27a \\
& A_27b).((p \ (ap \ (ap \ (ap \ (c_2Epath_2EfirstP_at \ A_27a \ A_27b) \ V3P) \\
& (ap \ (ap \ (ap \ (c_2Epath_2Epcns \ A_27a \ A_27b) \ V5x) \ V6r) \ V7p)) \ V4n)) \Leftrightarrow \\
& (((V4n = c_2Enum_2E0) \wedge (p \ (ap \ V3P \ V5x))) \vee ((p \ (ap \ (ap \ c_2Eprim_rec_2E_3C \\
& c_2Enum_2E0) \ V4n)) \wedge ((\neg (p \ (ap \ V3P \ V5x))) \wedge (p \ (ap \ (ap \ (ap \ (c_2Epath_2EfirstP_at \\
& A_27a \ A_27b) \ V3P) \ V7p) \ (ap \ (ap \ c_2Earithmetic_2E_2D \ V4n) \ (ap \ c_2Earithmetic_2ENUMERAL \\
& (ap \ c_2Earithmetic_2EBIT1 \ c_2Earithmetic_2EZERO))))))))))
\end{aligned} \tag{39}$$

Assume the following.

$$(\forall V0n \in ty_2Enum_2Enum.(\neg (p \ (ap \ (ap \ c_2Eprim_rec_2E_3C \ V0n) \ c_2Enum_2E0)))) \tag{40}$$

Theorem 1

$$\begin{aligned} & \forall A_{27a}. \text{nonempty } A_{27a} \Rightarrow \forall A_{27b}. \text{nonempty } A_{27b} \Rightarrow (\\ & \quad \forall V0P \in (2^{A_{27a}}). (\forall V1p \in (\text{ty_2Epath_2Epath } A_{27a} \\ & \quad A_{27b}). ((p \text{ (ap (ap (ap (c_2Epath_2EfirstP_at } A_{27a} A_{27b}) V0P) \\ & \quad V1p) c_2Enum_2E0})) \Leftrightarrow (p \text{ (ap } V0P \text{ (ap (c_2Epath_2Efirst } A_{27a} A_{27b}) \\ & \quad V1p)))))) \end{aligned}$$