

thm\_2Epath\_2Efirst\_\_plink  
(TMJz6sfWCVWANUbyEmq1zRGxuXVCBou87BZ)

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**Definition 1** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Ebool\_2ET$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 3** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 4** We define  $c\_2Ebool\_2EF$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 5** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 6** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2EF$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{1}$$

Let  $ty\_2Ellist\_2Ellist : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty\_2Ellist\_2Ellist A0) \tag{2}$$

Let  $ty\_2Epath\_2Epath : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epath\_2Epath A0 A1) \tag{3}$$

Let  $c\_2Epath\_2EfromPath : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epath\_2EfromPath A\_27a A\_27b \in ((ty\_2Epair\_2Eprod A\_27a (ty\_2Ellist\_2Ellist (ty\_2Epair\_2Eprod A\_27b A\_27a)))^{(ty\_2Epath\_2Epath A\_27a A\_27b)}) \tag{4}$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epair\_2EFST A\_27a A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod A\_27a A\_27b)}) \tag{5}$$

**Definition 7** We define  $c\_2Epath\_2Efirst$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b)$

Let  $c\_2Epath\_2Elast : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epath\_2Elast \\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epath\_2Epath\ A\_27a\ A\_27b)}) \end{aligned} \quad (6)$$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND \\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \end{aligned} \quad (7)$$

**Definition 8** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2))$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod \\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \end{aligned} \quad (8)$$

**Definition 9** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1y \in A\_27b. (ap\ (c\_2Epair\_2EABS\_prod\ x\ y))$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (9)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (10)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (11)$$

**Definition 10** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

**Definition 11** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (12)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (13)$$

**Definition 12** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum. (ap\ c\_2Enum\_2EABS\_num\ m)$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (14)$$

**Definition 13** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap (ap c\_2Earithmetic$

**Definition 14** We define  $c\_2Earithmetic\_2ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Earithmetic\_2E\_2D : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2D \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (15)$$

Let  $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty\_2Eoption\_2Eoption A0) \quad (16)$$

Let  $c\_2Ellist\_2Ellist\_rep : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ellist\_2Ellist\_rep A\_27a \in ((ty\_2Eoption\_2Eoption A\_27a)^{ty\_2Enum\_2Enum})^{(ty\_2Ellist\_2Ellist A\_27a)} \quad (17)$$

Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty ty\_2Eone\_2Eone \quad (18)$$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Esum\_2Esum A0 A1) \quad (19)$$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Esum\_2EABS\_sum A\_27a A\_27b \in ((ty\_2Esum\_2Esum A\_27a A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \quad (20)$$

**Definition 15** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap (c\_2Esum\_2EABS$

Let  $c\_2Eoption\_2Eoption\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Eoption\_2Eoption\_ABS A\_27a \in ((ty\_2Eoption\_2Eoption A\_27a)^{(ty\_2Esum\_2Esum A\_27a ty\_2Eone\_2Eone)}) \quad (21)$$

**Definition 16** We define  $c\_2Eoption\_2ESOME$  to be  $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.(ap (c\_2Eoption\_2Eoption$

**Definition 17** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.\mathbf{if} (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge P x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 18** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Let  $c\_2Ellist\_2Ellist\_abs : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow c\_2Ellist\_2Ellist\_abs A\_27a \in ((ty\_2Ellist\_2Ellist A\_27a)^{(ty\_2Eoption\_2Eoption A\_27a)^{ty\_2Enum\_2Enum}}) \quad (22)$$

**Definition 19** We define  $c\_2Ellist\_2ELCONS$  to be  $\lambda A\_27a : \iota.\lambda V0h \in A\_27a.\lambda V1t \in (ty\_2Ellist\_2Ellist A\_27a)$ .  
Let  $c\_2Epath\_2EtoPath : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epath\_2EtoPath \\ A\_27a A\_27b \in & ((ty\_2Epath\_2Epath A\_27a A\_27b)^{(ty\_2Epair\_2Eprod A\_27a (ty\_2Ellist\_2Ellist (ty\_2Epair\_2Eprod \\ & (23) \end{aligned}$$

**Definition 20** We define  $c\_2Epath\_2Epcons$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1r \in A\_27b.\lambda V2p$

**Definition 21** We define  $c\_2Eone\_2Eone$  to be  $(ap (c\_2Emin\_2E40 ty\_2Eone\_2Eone) (\lambda V0x \in ty\_2Eone\_2Eone$

**Definition 22** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap (c\_2Esum\_2EABS$

**Definition 23** We define  $c\_2Eoption\_2ENONE$  to be  $\lambda A\_27a : \iota.(ap (c\_2Eoption\_2Eoption\_ABS A\_27a) (c$

**Definition 24** We define  $c\_2Ellist\_2ELNIL$  to be  $\lambda A\_27a : \iota.(ap (c\_2Ellist\_2Ellist\_abs A\_27a) (\lambda V0n \in ty$

**Definition 25** We define  $c\_2Epath\_2Estopped\_at$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.(ap (c\_2Epath$

Let  $c\_2Epath\_2Eplink : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epath\_2Eplink \\ A\_27a A\_27b \in & (((ty\_2Epath\_2Epath A\_27a A\_27b)^{(ty\_2Epath\_2Epath A\_27a A\_27b)})^{(ty\_2Epath\_2Epath A\_27a A\_27b)} \\ & (24) \end{aligned}$$

Assume the following.

$$True \quad (25)$$

Assume the following.

$$\begin{aligned} & (\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow \\ & True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (( \\ & (p V0t) \Rightarrow False) \Leftrightarrow \neg (p V0t)))))) \end{aligned} \quad (26)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (28)$$

Assume the following.

$$\begin{aligned} & (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow \\ & ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & (\forall V0x \in 2.(\forall V1x\_27 \in 2.(\forall V2y \in 2.(\forall V3y\_27 \in \\ & 2.(((p V0x) \Leftrightarrow (p V1x\_27)) \wedge ((p V1x\_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y\_27)))))) \Rightarrow \\ & (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x\_27) \Rightarrow (p V3y\_27)))))) \end{aligned} \quad (30)$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\
& \quad \forall V0P \in (2^{(ty\_2Epath\_2Epath\ A\_27a\ A\_27b)}).((\forall V1p \in \\
& \quad (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).(p\ (ap\ V0P\ V1p))) \Leftrightarrow ((\forall V2x \in \\
& \quad A\_27a.(p\ (ap\ V0P\ (ap\ (c\_2Epath\_2Estoppped\_at\ A\_27a\ A\_27b)\ V2x)))) \wedge \\
& \quad (\forall V3x \in A\_27a.(\forall V4r \in A\_27b.(\forall V5p \in (ty\_2Epath\_2Epath \\
& \quad A\_27a\ A\_27b).(p\ (ap\ V0P\ (ap\ (ap\ (ap\ (c\_2Epath\_2Epcons\ A\_27a\ A\_27b) \\
& \quad V3x)\ V4r)\ V5p))))))))))
\end{aligned} \tag{31}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\
& (\forall V0x \in A\_27a.((ap\ (c\_2Epath\_2Efirst\ A\_27a\ A\_27b)\ (ap\ (c\_2Epath\_2Estoppped\_at \\
& \quad A\_27a\ A\_27b)\ V0x)) = V0x)) \wedge (\forall V1x \in A\_27a.(\forall V2r \in A\_27b. \\
& \quad (\forall V3p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).((ap\ (c\_2Epath\_2Efirst \\
& \quad A\_27a\ A\_27b)\ (ap\ (ap\ (ap\ (c\_2Epath\_2Epcons\ A\_27a\ A\_27b)\ V1x)\ V2r) \\
& \quad V3p)) = V1x))))))
\end{aligned} \tag{32}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\
& (\forall V0x \in A\_27a.((ap\ (c\_2Epath\_2Elast\ A\_27a\ A\_27b)\ (ap\ (c\_2Epath\_2Estoppped\_at \\
& \quad A\_27a\ A\_27b)\ V0x)) = V0x)) \wedge (\forall V1x \in A\_27a.(\forall V2r \in A\_27b. \\
& \quad (\forall V3p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).((ap\ (c\_2Epath\_2Elast \\
& \quad A\_27a\ A\_27b)\ (ap\ (ap\ (ap\ (c\_2Epath\_2Epcons\ A\_27a\ A\_27b)\ V1x)\ V2r) \\
& \quad V3p)) = (ap\ (c\_2Epath\_2Elast\ A\_27a\ A\_27b)\ V3p))))))
\end{aligned} \tag{33}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\
& (\forall V0x \in A\_27a.(\forall V1p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b). \\
& ((ap\ (ap\ (c\_2Epath\_2Eplink\ A\_27a\ A\_27b)\ (ap\ (c\_2Epath\_2Estoppped\_at \\
& \quad A\_27a\ A\_27b)\ V0x))\ V1p) = V1p))) \wedge (\forall V2x \in A\_27a.(\forall V3r \in \\
& \quad A\_27b.(\forall V4p1 \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).(\forall V5p2 \in \\
& \quad (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).((ap\ (ap\ (c\_2Epath\_2Eplink\ A\_27a \\
& \quad A\_27b)\ (ap\ (ap\ (ap\ (c\_2Epath\_2Epcons\ A\_27a\ A\_27b)\ V2x)\ V3r)\ V4p1)) \\
& \quad V5p2) = (ap\ (ap\ (ap\ (c\_2Epath\_2Epcons\ A\_27a\ A\_27b)\ V2x)\ V3r)\ (ap\ ( \\
& \quad ap\ (c\_2Epath\_2Eplink\ A\_27a\ A\_27b)\ V4p1)\ V5p2))))))
\end{aligned} \tag{34}$$

**Theorem 1**

$$\begin{aligned}
& \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\
& \quad \forall V0p1 \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).(\forall V1p2 \in ( \\
& \quad ty\_2Epath\_2Epath\ A\_27a\ A\_27b).(((ap\ (c\_2Epath\_2Elast\ A\_27a\ A\_27b) \\
& \quad V0p1) = (ap\ (c\_2Epath\_2Efirst\ A\_27a\ A\_27b)\ V1p2)) \Rightarrow ((ap\ (c\_2Epath\_2Efirst \\
& \quad A\_27a\ A\_27b)\ (ap\ (ap\ (c\_2Epath\_2Eplink\ A\_27a\ A\_27b)\ V0p1)\ V1p2)) = \\
& \quad (ap\ (c\_2Epath\_2Efirst\ A\_27a\ A\_27b)\ V0p1))))))
\end{aligned}$$