

thm\_2Epath\_2Efirst\_\_take  
(TMXy8RDdKC7jUXCsLzYCCQPCU5w7kEtk1Pg)

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**Definition 1** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.$ **if**  $(\exists x \in A.p (ap P x))$  **then** *(the*  $(\lambda x.x \in A \wedge p)$  *of type*  $\iota \Rightarrow \iota$ .

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  *of type*  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2E\_3F$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap V0P (ap (c\_2Emin\_2E\_40 A\_27a P))$

**Definition 4** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  *of type*  $\iota$ .

**Definition 5** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 6** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a} P))$

**Definition 7** We define  $c\_2Ebool\_2E\_5C\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $ty\_2Epath\_2Epath : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epath\_2Epath A0 A1) \tag{1}$$

Let  $c\_2Epath\_2Etail : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epath\_2Etail A\_27a A\_27b \in ((ty\_2Epath\_2Epath A\_27a A\_27b)^{(ty\_2Epath\_2Epath A\_27a A\_27b)}) \tag{2}$$

Let  $c\_2Epath\_2Efirst\_label : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty A\_27a \Rightarrow \forall A\_27b.nonempty A\_27b \Rightarrow c\_2Epath\_2Efirst\_label A\_27a A\_27b \in (A\_27b^{(ty\_2Epath\_2Epath A\_27a A\_27b)}) \tag{3}$$

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty\_2Epair\_2Eprod A0 A1) \tag{4}$$

Let  $ty\_2Ellist\_2Ellist : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Ellist\_2Ellist\ A0) \quad (5)$$

Let  $c\_2Epath\_2EfromPath : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epath\_2EfromPath\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ (ty\_2Ellist\_2Ellist\ (ty\_2Epair\_2Eprod\ A\_27b\ A\_27a)))^{(ty\_2Epath\_2Epath\ A\_27a\ A\_27b)}) \quad (6)$$

Let  $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b)^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)} \quad (7)$$

Let  $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a)^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)} \quad (8)$$

**Definition 8** We define  $c\_2Epath\_2Efirst$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b)$

**Definition 9** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c\_2Ebool\_2E\_21\ 2)\ (\lambda V2t \in 2. \dots))))$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (9)$$

**Definition 10** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1y \in A\_27b. (ap\ (c\_2Epair\_2E\_2C\ x\ y))$

Let  $c\_2Enum\_2EZERO\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2EZERO\_REP \in \omega \quad (10)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \quad (11)$$

Let  $c\_2Enum\_2EABS\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EABS\_num \in (ty\_2Enum\_2Enum^{\omega}) \quad (12)$$

**Definition 11** We define  $c\_2Enum\_2E0$  to be  $(ap\ c\_2Enum\_2EABS\_num\ c\_2Enum\_2EZERO\_REP)$ .

**Definition 12** We define  $c\_2Earithmetic\_2EZERO$  to be  $c\_2Enum\_2E0$ .

Let  $c\_2Enum\_2EREP\_num : \iota$  be given. Assume the following.

$$c\_2Enum\_2EREP\_num \in (\omega^{ty\_2Enum\_2Enum}) \quad (13)$$

Let  $c\_2Enum\_2ESUC\_REP : \iota$  be given. Assume the following.

$$c\_2Enum\_2ESUC\_REP \in (\omega^{\omega}) \quad (14)$$

**Definition 13** We define  $c\_2Enum\_2ESUC$  to be  $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_num$

Let  $c\_2Earithmetic\_2E\_2B : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (15)$$

**Definition 14** We define  $c\_2Earithmetic\_2EBIT1$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

**Definition 15** We define  $c\_2Earithmetic\_2ENUMERAL$  to be  $\lambda V0x \in ty\_2Enum\_2Enum.V0x$ .

Let  $c\_2Earithmetic\_2E\_2D : \iota$  be given. Assume the following.

$$c\_2Earithmetic\_2E\_2D \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (16)$$

Let  $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Eoption\_2Eoption\ A0) \quad (17)$$

Let  $c\_2Ellist\_2Ellist\_rep : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ellist\_2Ellist\_rep\ A\_27a \in \\ (((ty\_2Eoption\_2Eoption\ A\_27a)^{ty\_2Enum\_2Enum})^{(ty\_2Ellist\_2Ellist\ A\_27a)}) \quad (18)$$

Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \quad (19)$$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \quad (20)$$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \quad (21)$$

**Definition 16** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap\ (c\_2Esum\_2EABS$

Let  $c\_2Eoption\_2Eoption\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Eoption\_2Eoption\_ABS\ A\_27a \in ((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ ty\_2Eone\_2Eone)}) \quad (22)$$

**Definition 17** We define  $c\_2Eoption\_2ESOME$  to be  $\lambda A\_27a : \iota. \lambda V0x \in A\_27a. (ap\ (c\_2Eoption\_2Eoption\_ABS\ A\_27a)\ V0x)$

**Definition 18** We define  $c\_2Ebool\_2EF$  to be  $(ap\ (c\_2Ebool\_2E21\ 2)\ (\lambda V0t \in 2.V0t))$ .

**Definition 19** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A\_27a. (\lambda V2t2 \in A\_27a. (ap\ (c\_2Ebool\_2E21\ 2)\ V2t2)\ V1t1)\ V0t))$

Let  $c\_2Ellist\_2Ellist\_abs : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ellist\_2Ellist\_abs\ A\_27a \in ((ty\_2Ellist\_2Ellist\ A\_27a)^{(ty\_2Eoption\_2Eoption\ A\_27a)^{ty\_2Enum\_2Enum}}) \quad (23)$$

**Definition 20** We define  $c\_2Ellist\_2ELCONS$  to be  $\lambda A\_27a : \iota. \lambda V0h \in A\_27a. \lambda V1t \in (ty\_2Ellist\_2Ellist\ A\_27a)\ V0h. (ap\ (c\_2Ellist\_2Ellist\_abs\ A\_27a)\ V1t)$

Let  $c\_2Epath\_2EtoPath : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epath\_2EtoPath\ A\_27a\ A\_27b \in ((ty\_2Epath\_2Epath\ A\_27a\ A\_27b)^{(ty\_2Epair\_2Eprod\ A\_27a\ (ty\_2Ellist\_2Ellist\ (ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)))}) \quad (24)$$

**Definition 21** We define  $c\_2Epath\_2Epcns$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. \lambda V1r \in A\_27b. \lambda V2p \in A\_27b. (ap\ (c\_2Epath\_2EtoPath\ A\_27a\ A\_27b)\ V2p)\ V1r)\ V0x)$

**Definition 22** We define  $c\_2Eone\_2Eone$  to be  $(ap\ (c\_2Emin\_2E40\ ty\_2Eone\_2Eone)\ (\lambda V0x \in ty\_2Eone\_2Eone.V0x))$

**Definition 23** We define  $c\_2Ebool\_2E7E$  to be  $(\lambda V0t \in 2. (ap\ (ap\ c\_2Emin\_2E3D\_3D\_3E\ V0t)\ c\_2Ebool\_2E7E)\ V0t))$

**Definition 24** We define  $c\_2Esum\_2EINR$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0e \in A\_27b. (ap\ (c\_2Esum\_2EABS\ A\_27a\ A\_27b)\ V0e)$

**Definition 25** We define  $c\_2Eoption\_2ENONE$  to be  $\lambda A\_27a : \iota. (ap\ (c\_2Eoption\_2Eoption\_ABS\ A\_27a)\ (c\_2Eoption\_2ENONE))$

**Definition 26** We define  $c\_2Ellist\_2ELNIL$  to be  $\lambda A\_27a : \iota. (ap\ (c\_2Ellist\_2Ellist\_abs\ A\_27a)\ (\lambda V0n \in ty\_2Ellist\_2Ellist\ A\_27a.V0n))$

**Definition 27** We define  $c\_2Epath\_2Estopped\_at$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0x \in A\_27a. (ap\ (c\_2Epath\_2EtoPath\ A\_27a\ A\_27b)\ V0x)$

Let  $c\_2Epath\_2Etake : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epath\_2Etake\ A\_27a\ A\_27b \in (((ty\_2Epath\_2Epath\ A\_27a\ A\_27b)^{(ty\_2Epath\_2Epath\ A\_27a\ A\_27b)})^{ty\_2Enum\_2Enum}) \quad (25)$$

Assume the following.

$$(\forall V0m \in ty\_2Enum\_2Enum. ((V0m = c\_2Enum\_2E0) \vee (\exists V1n \in ty\_2Enum\_2Enum. (V0m = (ap\ c\_2Enum\_2ESUC\ V1n)))))) \quad (26)$$

Assume the following.

$$True \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a. ((V0x = V0x) \Leftrightarrow True)) \quad (28)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & (\forall V0x \in A\_27a. ((ap\ (c\_2Epath\_2Efirst\ A\_27a\ A\_27b)\ (ap\ (c\_2Epath\_2Estopped\_at \\ & \quad A\_27a\ A\_27b)\ V0x)) = V0x)) \wedge (\forall V1x \in A\_27a. (\forall V2r \in A\_27b. \\ & \quad (\forall V3p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b). ((ap\ (c\_2Epath\_2Efirst \\ & \quad A\_27a\ A\_27b)\ (ap\ (ap\ (ap\ (c\_2Epath\_2Epcons\ A\_27a\ A\_27b)\ V1x)\ V2r) \\ & \quad \quad V3p)) = V1x)))))) \end{aligned} \quad (29)$$

Assume the following.

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & (\forall V0p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b). ((ap\ (ap\ (c\_2Epath\_2Etake \\ & \quad A\_27a\ A\_27b)\ c\_2Enum\_2E0)\ V0p) = (ap\ (c\_2Epath\_2Estopped\_at\ A\_27a \\ & \quad A\_27b)\ (ap\ (c\_2Epath\_2Efirst\ A\_27a\ A\_27b)\ V0p)))) \wedge (\forall V1n \in \\ & \quad ty\_2Enum\_2Enum. (\forall V2p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b). \\ & \quad ((ap\ (ap\ (c\_2Epath\_2Etake\ A\_27a\ A\_27b)\ (ap\ c\_2Enum\_2ESUC\ V1n)) \\ & \quad V2p) = (ap\ (ap\ (ap\ (c\_2Epath\_2Epcons\ A\_27a\ A\_27b)\ (ap\ (c\_2Epath\_2Efirst \\ & \quad A\_27a\ A\_27b)\ V2p))\ (ap\ (c\_2Epath\_2Efirst\_label\ A\_27a\ A\_27b)\ V2p)) \\ & \quad (ap\ (ap\ (c\_2Epath\_2Etake\ A\_27a\ A\_27b)\ V1n)\ (ap\ (c\_2Epath\_2Etail \\ & \quad \quad A\_27a\ A\_27b)\ V2p)))))) \end{aligned} \quad (30)$$

**Theorem 1**

$$\begin{aligned} & \forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ( \\ & \forall V0p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b). (\forall V1i \in ty\_2Enum\_2Enum. \\ & ((ap\ (c\_2Epath\_2Efirst\ A\_27a\ A\_27b)\ (ap\ (ap\ (c\_2Epath\_2Etake\ A\_27a \\ & \quad A\_27b)\ V1i)\ V0p)) = (ap\ (c\_2Epath\_2Efirst\ A\_27a\ A\_27b)\ V0p)))) \end{aligned}$$