thm_2Epath_2Epconcat__thm (TMW-muS1g9bQdgRTLDsFEPGDiKv2GR8cYL2S)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 3 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 4 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{1}$$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{2}$$

Let $ty\_2Eone\_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \tag{3}$$

Definition 5 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 6 *We define* c_2Eone_2Eone *to be* $(ap\ (\mathsf{c\_2Emin\_2E\_40}\ ty\_2Eone\_2Eone)\ (\lambda V0x \in ty\_2Eone\_2E$

Definition 7 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 8 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2EF}$

Definition 9 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $ty\_2Esum\_2Esum : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0{\Rightarrow}\forall A1.nonempty\ A1{\Rightarrow}nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \tag{4}$$

Let $c\_2Esum\_2EABS\_\_sum : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}\forall A\_27b.nonempty\ A\_27b{\Rightarrow}c\_2Esum\_2EABS\_\_sum\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{(((2^{A\_27b})^{A\_27a})^{2})}) \tag{5}$$

Definition 10 *We define* c_2Esum_2EINR *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27b.(ap\ (c\_2Esum\_2EABS$

Let $ty\_2Eoption\_2Eoption : \iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0{\Rightarrow}nonempty\ (ty\_2Eoption\_2Eoption\ A0) \tag{6}$$

Let $c\_2Eoption\_2Eoption\_\_ABS : \iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}c\_2Eoption\_2Eoption\_\_ABS\ A\_27a \in ((ty\_2Eoption\_2Eoption\ A\_27a)^{(ty\_2Esum\_2Esum\ A\_27a\ ty\_2Eone\_2Eone)}) \tag{7}$$

Definition 11 *We define* c_2Eoption_2ENONE *to be* $\lambda A\_27a : \iota.(ap\ (c\_2Eoption\_2Eoption\_\_ABS\ A\_27a)\ ($

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{8}$$

Let $ty\_2Ellist\_2Ellist : \iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0{\Rightarrow}nonempty\ (ty\_2Ellist\_2Ellist\ A0) \tag{9}$$

Let $c\_2Ellist\_2Ellist\_\_abs : \iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}c\_2Ellist\_2Ellist\_\_abs\ A\_27a \in ((ty\_2Ellist\_2Ellist\ A\_27a)^{((ty\_2Eoption\_2Eoption\ A\_27a)^{ty\_2Enum\_2Enum})}) \tag{10}$$

Definition 12 *We define* c_2Ellist_2ELNIL *to be* $\lambda A\_27a : \iota.(ap\ (c\_2Ellist\_2Ellist\_\_abs\ A\_27a)\ (\lambda V0n \in ty\_$

Let $ty\_2Epath\_2Epath : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0{\Rightarrow}\forall A1.nonempty\ A1{\Rightarrow}nonempty\ (ty\_2Epath\_2Epath\ A0\ A1) \tag{11}$$

Let $c\_2Epath\_2EfromPath : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}\forall A\_27b.nonempty\ A\_27b{\Rightarrow}c\_2Epath\_2EfromPath\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ (ty\_2Ellist\_2Ellist\ (ty\_2Epair\_2Eprod\ A\_27b\ A\_27a)))^{(ty\_2Epath\_2Epath\ A\_27a\ A\_27b)}) \tag{12}$$

Let $c\_2Epair\_2ESND : \iota{\Rightarrow}\iota{\Rightarrow}\iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a{\Rightarrow}\forall A\_27b.nonempty\ A\_27b{\Rightarrow}c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{13}$$

Definition 13 *We define* c_2Epath_2Efirst *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0p \in (ty\_2Epath\_2Epath\ A\_27a\ A$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \quad (14)$$

Definition 14 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2$

Let $c\_2Enum\_2EZERO\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2EZERO\_\_REP \in omega \quad (15)$$

Let $c\_2Enum\_2EABS\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EABS\_\_num \in (ty\_2Enum\_2Enum^{omega}) \quad (16)$$

Definition 15 *We define* c_2Enum_2E0 *to be* $(ap\ c\_2Enum\_2EABS\_\_num\ c\_2Enum\_2EZERO\_\_REP)$.

Definition 16 *We define* c_2Earithmetic_2EZERO *to be* c_2Enum_2E0.

Let $c\_2Enum\_2EREP\_\_num : \iota$ be given. Assume the following.

$$c\_2Enum\_2EREP\_\_num \in (omega^{ty\_2Enum\_2Enum}) \quad (17)$$

Let $c\_2Enum\_2ESUC\_\_REP : \iota$ be given. Assume the following.

$$c\_2Enum\_2ESUC\_\_REP \in (omega^{omega}) \quad (18)$$

Definition 17 *We define* c_2Enum_2ESUC *to be* $\lambda V0m \in ty\_2Enum\_2Enum.(ap\ c\_2Enum\_2EABS\_\_num$

Let $c\_2Earithmetic\_2E\_2B : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2B \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (19)$$

Definition 18 *We define* c_2Earithmetic_2EBIT1 *to be* $\lambda V0n \in ty\_2Enum\_2Enum.(ap\ (ap\ c\_2Earithmetic$

Definition 19 *We define* c_2Earithmetic_2ENUMERAL *to be* $\lambda V0x \in ty\_2Enum\_2Enum.V0x$.

Let $c\_2Earithmetic\_2E\_2D : \iota$ be given. Assume the following.

$$c\_2Earithmetic\_2E\_2D \in ((ty\_2Enum\_2Enum^{ty\_2Enum\_2Enum})^{ty\_2Enum\_2Enum}) \quad (20)$$

Let $c\_2Ellist\_2Ellist\_\_rep : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ellist\_2Ellist\_\_rep\ A\_27a \in (((ty\_2Eoption\_2Eoption\ A\_27a)^{ty\_2Enum\_2Enum})^{(ty\_2Ellist\_2Ellist\ A\_27a)}) \quad (21)$$

Definition 20 *We define* c_2Esum_2EINL *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0e \in A\_27a.(ap\ (c\_2Esum\_2EABS$

Definition 21 *We define* c_2Eoption_2ESOME *to be* $\lambda A\_27a : \iota.\lambda V0x \in A\_27a.(ap\ (c\_2Eoption\_2Eoption\_\_$

Definition 22 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Definition 23 *We define* c_2Ellist_2ELCONS *to be* $\lambda A\_27a : \iota.\lambda V0h \in A\_27a.\lambda V1t \in (ty\_2Ellist\_2Ellist\ A.$

Let $c\_2Epath\_2EtoPath : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epath\_2EtoPath\ A\_27a\ A\_27b \in ((ty\_2Epath\_2Epath\ A\_27a\ A\_27b)^{(ty\_2Epair\_2Eprod\ A\_27a\ (ty\_2Ellist\_2Ellist\ (ty\_2Epair\_2Eprod} \quad (22)$$

Definition 24 *We define* c_2Epath_2Epcons *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1r \in A\_27b.\lambda V2p$

Definition 25 *We define* c_2Epath_2Estopped__at *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.(ap\ (c\_2Epath$

Let $c\_2Ellist\_2ELAPPEND : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ellist\_2ELAPPEND\ A\_27a \in (((ty\_2Ellist\_2Ellist\ A\_27a)^{(ty\_2Ellist\_2Ellist\ A\_27a)})^{(ty\_2Ellist\_2Ellist\ A\_27a)}) \quad (23)$$

Definition 26 *We define* c_2Epath_2Epconcat *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0p1 \in (ty\_2Epath\_2Epath\ A\_2$

Assume the following.

$$True \quad (24)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge ((((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge ((((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t))))))) \quad (25)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow True)) \quad (26)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0h1 \in A\_27a.(\forall V1t1 \in (ty\_2Ellist\_2Ellist\ A\_27a).(\forall V2h2 \in A\_27a.(\forall V3t2 \in (ty\_2Ellist\_2Ellist\ A\_27a).(((ap\ (ap\ (\mathsf{c\_2Ellist\_2ELCONS}\ A\_27a)\ V0h1)\ V1t1) = (ap\ (ap\ (\mathsf{c\_2Ellist\_2ELCONS}\ A\_27a)\ V2h2)\ V3t2)) \Leftrightarrow ((V0h1 = V2h2) \wedge (V1t1 = V3t2))))))) \quad (27)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow ((\forall V0x \in (ty\_2Ellist\_2Ellist\ A\_27a).((ap\ (ap\ (c\_2Ellist\_2ELAPPEND\ A\_27a)\ (\mathsf{c\_2Ellist\_2ELNIL}\ A\_27a))\ V0x) = V0x)) \wedge (\forall V1h \in A\_27a.(\forall V2t \in (ty\_2Ellist\_2Ellist\ A\_27a).(\forall V3x \in (ty\_2Ellist\_2Ellist\ A\_27a).((ap\ (ap\ (c\_2Ellist\_2ELAPPEND\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Ellist\_2ELCONS}\ A\_27a)\ V1h)\ V2t))\ V3x) = (ap\ (ap\ (\mathsf{c\_2Ellist\_2ELCONS}\ A\_27a)\ V1h)\ (ap\ (ap\ (c\_2Ellist\_2ELAPPEND\ A\_27a)\ V2t)\ V3x)))))))) \quad (28)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27b.(\forall V2a \in A\_27a.(\forall V3b \in A\_27b.(((ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27a\ A\_27b)\ V0x)\ V1y) = (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27a\ A\_27b)\ V2a)\ V3b)) \Leftrightarrow ((V0x = V2a) \wedge (V1y = V3b))))))) \quad (29)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27b.((ap\ (c\_2Epair\_2EFST\ A\_27a\ A\_27b)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27a\ A\_27b)\ V0x)\ V1y)) = V0x))) \quad (30)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in A\_27b.((ap\ (c\_2Epair\_2ESND\ A\_27a\ A\_27b)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27a\ A\_27b)\ V0x)\ V1y)) = V1y))) \quad (31)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ((\forall V0a \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).((ap\ (c\_2Epath\_2EtoPath\ A\_27a\ A\_27b)\ (ap\ (c\_2Epath\_2EfromPath\ A\_27a\ A\_27b)\ V0a)) = V0a)) \wedge (\forall V1r \in (ty\_2Epair\_2Eprod\ A\_27a\ (ty\_2Ellist\_2Ellist\ (ty\_2Epair\_2Eprod\ A\_27b\ A\_27a))).((ap\ (c\_2Epath\_2EfromPath\ A\_27a\ A\_27b)\ (ap\ (c\_2Epath\_2EtoPath\ A\_27a\ A\_27b)\ V1r)) = V1r))) \quad (32)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow (\forall V0r \in (ty\_2Epair\_2Eprod\ A\_27a\ (ty\_2Ellist\_2Ellist\ (ty\_2Epair\_2Eprod\ A\_27b\ A\_27a))).(\forall V1r\_27 \in (ty\_2Epair\_2Eprod\ A\_27a\ (ty\_2Ellist\_2Ellist\ (ty\_2Epair\_2Eprod\ A\_27b\ A\_27a))).(((ap\ (c\_2Epath\_2EtoPath\ A\_27a\ A\_27b)\ V0r) = (ap\ (c\_2Epath\_2EtoPath\ A\_27a\ A\_27b)\ V1r\_27)) \Leftrightarrow (V0r = V1r\_27)))) \quad (33)$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow ((\forall V0x \in A\_27a.((ap\ (\mathsf{c\_2Epath\_2Efirst}\ A\_27a\ A\_27b)\ (ap\ (\mathsf{c\_2Epath\_2Estopped\_\_at}\ A\_27a\ A\_27b)\ V0x)) = V0x)) \wedge (\forall V1x \in A\_27a.(\forall V2r \in A\_27b. (\forall V3p \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).((ap\ (\mathsf{c\_2Epath\_2Efirst}\ A\_27a\ A\_27b)\ (ap\ (ap\ (ap\ (\mathsf{c\_2Epath\_2Epcons}\ A\_27a\ A\_27b)\ V1x)\ V2r)\ V3p)) = V1x))))) \tag{34}$$

Theorem 1

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c. nonempty\ A\_27c \Rightarrow \forall A\_27d.nonempty\ A\_27d \Rightarrow ((\forall V0x \in A\_27a. (\forall V1lab \in A\_27b.(\forall V2p2 \in (ty\_2Epath\_2Epath\ A\_27a\ A\_27b).((ap\ (ap\ (ap\ (\mathsf{c\_2Epath\_2Epconcat}\ A\_27a\ A\_27b)\ (ap\ (\mathsf{c\_2Epath\_2Estopped\_\_at}\ A\_27a\ A\_27b)\ V0x))\ V1lab)\ V2p2) = (ap\ (ap\ (ap\ (\mathsf{c\_2Epath\_2Epcons}\ A\_27a\ A\_27b)\ V0x)\ V1lab)\ V2p2))))) \wedge (\forall V3x \in A\_27c.(\forall V4r \in A\_27d.(\forall V5p \in (ty\_2Epath\_2Epath\ A\_27c\ A\_27d).(\forall V6lab \in A\_27d.(\forall V7p2 \in (ty\_2Epath\_2Epath\ A\_27c\ A\_27d).((ap\ (ap\ (ap\ (\mathsf{c\_2Epath\_2Epconcat}\ A\_27c\ A\_27d)\ (ap\ (ap\ (ap\ (\mathsf{c\_2Epath\_2Epcons}\ A\_27c\ A\_27d)\ V3x)\ V4r)\ V5p))\ V6lab)\ V7p2) = (ap\ (ap\ (ap\ (\mathsf{c\_2Epath\_2Epcons}\ A\_27c\ A\_27d)\ V3x)\ V4r)\ (ap\ (ap\ (ap\ (\mathsf{c\_2Epath\_2Epconcat}\ A\_27c\ A\_27d)\ V5p)\ V6lab)\ V7p2)))))))))$$