thm_2Epatricia_2EBRANCH__def (TML5nL3vmv9VMBL2S359RNDN9n9ESKRBRoP)

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Definition 1 *We define* c_2Emin_2E_3D *to be* $\lambda A.\lambda x \in A.\lambda y \in A.inj\_\_o\ (x = y)$ *of type* $\iota \Rightarrow \iota$.

Definition 2 *We define* c_2Emin_2E_3D_3D_3E *to be* $\lambda P \in 2.\lambda Q \in 2.inj\_\_o\ (p\ P \Rightarrow p\ Q)$ *of type* ι.

Definition 3 *We define* c_2Ebool_2ET *to be* $(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^2))\ (\lambda V0x \in 2.V0x))\ (\lambda V1x \in 2.V1$

Definition 4 *We define* c_2Ebool_2E_21 *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ (ap\ (\mathsf{c\_2Emin\_2E\_3D}\ (2^{A\_27}$

Definition 5 *We define* c_2Ebool_2E_2F_5C *to be* $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V2t \in 2$

Let $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \tag{1}$$

Let $c\_2Epair\_2EABS\_\_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})}) \tag{2}$$

Definition 6 *We define* c_2Epair_2E_2C *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2E$

Let $ty\_2Epatricia\_2Eptree : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty\_2Epatricia\_2Eptree\ A0) \tag{3}$$

Let $c\_2Epatricia\_2EEmpty : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epatricia\_2EEmpty\ A\_27a \in (ty\_2Epatricia\_2Eptree\ A\_27a) \tag{4}$$

Let $ty\_2Enum\_2Enum : \iota$ be given. Assume the following.

$$nonempty\ ty\_2Enum\_2Enum \tag{5}$$

Let $c\_2Epatricia\_2EBranch : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epatricia\_2EBranch\ A\_27a \in (((((ty\_2Epatricia\_2Eptree\ A\_27a)^{(ty\_2Epatricia\_2Eptree\ A\_27a)})^{(ty\_2Epatricia\_2Eptree\ A\_27a)})^{ty\_2Enum\_2Enum}) \tag{6}$$

Let $c\_2Epatricia\_2ELeaf : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epatricia\_2ELeaf\ A\_27a \in (((ty\_2Epatricia\_2Eptree\ A\_27a)^{A\_27a})^{ty\_2Enum\_2Enum}) \tag{7}$$

Definition 7 *We define* c_2Ecombin_2EK *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.(\lambda V0x \in A\_27a.(\lambda V1y \in A\_27b.V0x))$

Definition 8 *We define* c_2Ecombin_2ES *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.(\lambda V0f \in ((A\_27c^{A\_27b})^{A\_27a}$

Definition 9 *We define* c_2Ecombin_2EI *to be* $\lambda A\_27a : \iota.(ap\ (ap\ (\mathsf{c\_2Ecombin\_2ES}\ A\_27a\ (A\_27a^{A\_27a})\ A\_$

Let $c\_2Epatricia\_2Eptree\_\_CASE : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epat A\_27a\ A\_27b \in ((((A\_27b^{((((A\_27b^{(ty\_2Epatricia\_2Eptree\ A\_27a)})^{(ty\_2Epatricia\_2Eptree\ A\_27a)})^{ty\_2Enum\_2Enum})^{ty\_2Enum\_} \tag{8}$$

Let $c\_2Epair\_2ESND : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2ESND\ A\_27a\ A\_27b \in (A\_27b^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{9}$$

Let $c\_2Epair\_2EFST : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EFST\ A\_27a\ A\_27b \in (A\_27a^{(ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)}) \tag{10}$$

Definition 10 *We define* c_2Epair_2Epair__CASE *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda A\_27c : \iota.\lambda V0p \in (ty\_2Epair\_$

Definition 11 *We define* c_2Emin_2E_40 *to be* $\lambda A.\lambda P \in 2^A.$**if** $(\exists x \in A.p\ (ap\ P\ x))$ **then** $(the\ (\lambda x.x \in A \wedge p$ *of type* $\iota \Rightarrow \iota$.

Definition 12 *We define* c_2Ebool_2EF *to be* $(ap\ (\mathsf{c\_2Ebool\_2E\_21}\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 13 *We define* c_2Ebool_2E_7E *to be* $(\lambda V0t \in 2.(ap\ (ap\ \mathsf{c\_2Emin\_2E\_3D\_3D\_3E}\ V0t)\ \mathsf{c\_2Ebool\_2E}$

Definition 14 *We define* c_2Ebool_2E_3F *to be* $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap\ V0P\ (ap\ (\mathsf{c\_2Emin\_2E\_40}$

Definition 15 *We define* c_2Erelation_2EWF *to be* $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).(ap\ (\mathsf{c\_2Ebool\_2E\_21}$

Let $c\_2Ebool\_2EARB : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Ebool\_2EARB\ A\_27a \in A\_27a \tag{11}$$

Definition 16 *We define* c_2Ebool_2ECOND *to be* $\lambda A\_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A\_27a.(\lambda V2t2 \in A\_27a.($

Definition 17 *We define* c_2Erelation_2ERESTRICT *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0f \in (A\_27b^{A\_27a}).\lambda V1$

Definition 18 *We define* c_2Erelation_2ETC *to be* $\lambda A\_27a : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1a \in A\_27a.\lambda V2b$

Definition 19 *We define* c_2Erelation_2Eapprox *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M$

Definition 20 *We define* c_2Erelation_2Ethe__fun *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M$

Definition 21 *We define* c_2Erelation_2EWFREC *to be* $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0R \in ((2^{A\_27a})^{A\_27a}).\lambda V1M$

Definition 22 *We define* c_2Epatricia_2EBRANCH *to be* $\lambda A_27a : \iota.(ap\ (ap\ ($c_2Erelation_2EWFREC $(ty\_2E$

Definition 23 *We define* c_2Erelation_2EEMPTY__REL *to be* $\lambda A_27a : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27a.$c_2E

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0x \in A\_27a.((ap\ (\mathsf{c\_2Ecombin\_2EI}\ A\_27a)\ V0x) = V0x)) \tag{12}$$

Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow \forall A\_27c.nonempty\ A\_27c \Rightarrow (\forall V0x \in A\_27b.(\forall V1y \in A\_27c.(\forall V2f \in ((A\_27a^{A\_27c})^{A\_27b}).((ap\ (ap\ (\mathsf{c\_2Epair\_2Epair\_\_CASE}\ A\_27a\ A\_27b\ A\_27c)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ A\_27b\ A\_27c)\ V0x)\ V1y))\ V2f) = (ap\ (ap\ V2f\ V0x)\ V1y))))) \tag{13}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a{\Rightarrow}\forall A_27b.nonempty\ A_27b{\Rightarrow}(
(\forall V0v \in A_27b.(\forall V1f \in ((A_27b^{A_27a})^{ty_2Enum_2Enum}).
(\forall V2f1 \in ((((A_27b^{(ty_2Epatricia_2Eptree\ A_27a)})^{(ty_2Epatricia_2Eptree\ A_27a)})^{ty_2Enum_2Enum})^{ty_2Enum_2Enu}
((ap\ (ap\ (ap\ (ap\ (c_2Epatricia_2Eptree__CASE\ A_27a\ A_27b)\ (c_2Epatricia_2EEmpty\
A_27a))\ V0v)\ V1f)\ V2f1) = V0v)))){\wedge}((\forall V3a0 \in ty_2Enum_2Enum.
(\forall V4a1 \in A_27a.(\forall V5v \in A_27b.(\forall V6f \in ((A_27b^{A_27a})^{ty_2Enum_2Enum}).
(\forall V7f1 \in ((((A_27b^{(ty_2Epatricia_2Eptree\ A_27a)})^{(ty_2Epatricia_2Eptree\ A_27a)})^{ty_2Enum_2Enum})^{ty_2Enum_2Enu}
((ap\ (ap\ (ap\ (ap\ (c_2Epatricia_2Eptree__CASE\ A_27a\ A_27b)\ (ap\ (
ap\ (c_2Epatricia_2ELeaf\ A_27a)\ V3a0)\ V4a1))\ V5v)\ V6f)\ V7f1) = (ap\
(ap\ V6f\ V3a0)\ V4a1))))))){\wedge}(\forall V8a0 \in ty_2Enum_2Enum.(\forall V9a1 \in
ty_2Enum_2Enum.(\forall V10a2 \in (ty_2Epatricia_2Eptree\ A_27a).
(\forall V11a3 \in (ty_2Epatricia_2Eptree\ A_27a).(\forall V12v \in
A_27b.(\forall V13f \in ((A_27b^{A_27a})^{ty_2Enum_2Enum}).(\forall V14f1 \in
((((A_27b^{(ty_2Epatricia_2Eptree\ A_27a)})^{(ty_2Epatricia_2Eptree\ A_27a)})^{ty_2Enum_2Enum})^{ty_2Enum_2Enum}).
((ap\ (ap\ (ap\ (ap\ (c_2Epatricia_2Eptree__CASE\ A_27a\ A_27b)\ (ap\ (
ap\ (ap\ (ap\ (c_2Epatricia_2EBranch\ A_27a)\ V8a0)\ V9a1)\ V10a2)\ V11a3))
V12v)\ V13f)\ V14f1) = (ap\ (ap\ (ap\ (ap\ V14f1\ V8a0)\ V9a1)\ V10a2)\ V11a3)))))))))) \tag{14}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a{\Rightarrow}(p\ (ap\ (\mathsf{c_2Erelation_2EWF}\ A_27a)
(\mathsf{c_2Erelation_2EEMPTY__REL}\ A_27a))) \tag{15}$$

Assume the following.

$$\forall A_27a.nonempty\ A_27a{\Rightarrow}\forall A_27b.nonempty\ A_27b{\Rightarrow}(
\forall V0M \in ((A_27b^{A_27a})^{(A_27b^{A_27a})}).(\forall V1R \in ((2^{A_27a})^{A_27a}).
(\forall V2f \in (A_27b^{A_27a}).((V2f = (ap\ (ap\ (\mathsf{c_2Erelation_2EWFREC}
A_27a\ A_27b)\ V1R)\ V0M)){\Rightarrow}((p\ (ap\ (\mathsf{c_2Erelation_2EWF}\ A_27a)\ V1R)){\Rightarrow}
(\forall V3x \in A_27a.((ap\ V2f\ V3x) = (ap\ (ap\ V0M\ (ap\ (ap\ (ap\ (\mathsf{c_2Erelation_2ERESTRICT}
A_27a\ A_27b)\ V2f)\ V1R)\ V3x))\ V3x)))))))) \tag{16}$$

Theorem 1

$\forall A\_27a.nonempty\ A\_27a \Rightarrow (\forall V0p \in ty\_2Enum\_2Enum.($
$\forall V1m \in ty\_2Enum\_2Enum.(\forall V2t \in (ty\_2Epatricia\_2Eptree$
$A\_27a).(\forall V3v6 \in ty\_2Enum\_2Enum.(\forall V4v7 \in A\_27a.($
$\forall V5v8 \in ty\_2Enum\_2Enum.(\forall V6v9 \in ty\_2Enum\_2Enum.$
$(\forall V7v10 \in (ty\_2Epatricia\_2Eptree\ A\_27a).(\forall V8v11 \in$
$(ty\_2Epatricia\_2Eptree\ A\_27a).(\forall V9v12 \in ty\_2Enum\_2Enum.$
$(\forall V10v13 \in A\_27a.(\forall V11v24 \in ty\_2Enum\_2Enum.(\forall V12v25 \in$
$A\_27a.(\forall V13v26 \in ty\_2Enum\_2Enum.(\forall V14v27 \in ty\_2Enum\_2Enum.$
$(\forall V15v28 \in (ty\_2Epatricia\_2Eptree\ A\_27a).(\forall V16v29 \in$
$(ty\_2Epatricia\_2Eptree\ A\_27a).(\forall V17v14 \in ty\_2Enum\_2Enum.$
$(\forall V18v15 \in ty\_2Enum\_2Enum.(\forall V19v16 \in (ty\_2Epatricia\_2Eptree$
$A\_27a).(\forall V20v17 \in (ty\_2Epatricia\_2Eptree\ A\_27a).(\forall V21v36 \in$
$ty\_2Enum\_2Enum.(\forall V22v37 \in A\_27a.(\forall V23v38 \in ty\_2Enum\_2Enum.$
$(\forall V24v39 \in ty\_2Enum\_2Enum.(\forall V25v40 \in (ty\_2Epatricia\_2Eptree$
$A\_27a).(\forall V26v41 \in (ty\_2Epatricia\_2Eptree\ A\_27a).(((ap$
$(\mathsf{c\_2Epatricia\_2EBRANCH}\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Enum\_2Enum$
$(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree$
$A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))))\ V0p)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree\ A\_27a)$
$(ty\_2Epatricia\_2Eptree\ A\_27a)))\ V1m)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$(ty\_2Epatricia\_2Eptree\ A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))$
$(c\_2Epatricia\_2EEmpty\ A\_27a))\ V2t)))) = V2t) \wedge (((ap\ (\mathsf{c\_2Epatricia\_2EBRANCH}$
$A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod$
$ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree\ A\_27a)$
$(ty\_2Epatricia\_2Eptree\ A\_27a))))\ V0p)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree\ A\_27a)$
$(ty\_2Epatricia\_2Eptree\ A\_27a)))\ V1m)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$(ty\_2Epatricia\_2Eptree\ A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))$
$(ap\ (ap\ (c\_2Epatricia\_2ELeaf\ A\_27a)\ V3v6)\ V4v7))\ (c\_2Epatricia\_2EEmpty$
$A\_27a))))) = (ap\ (ap\ (c\_2Epatricia\_2ELeaf\ A\_27a)\ V3v6)\ V4v7)) \wedge$
$(((ap\ (\mathsf{c\_2Epatricia\_2EBRANCH}\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Enum\_2Enum$
$(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree$
$A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))))\ V0p)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree\ A\_27a)$
$(ty\_2Epatricia\_2Eptree\ A\_27a)))\ V1m)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$(ty\_2Epatricia\_2Eptree\ A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))$
$(ap\ (ap\ (ap\ (ap\ (c\_2Epatricia\_2EBranch\ A\_27a)\ V5v8)\ V6v9)\ V7v10)$
$V8v11))\ (c\_2Epatricia\_2EEmpty\ A\_27a))))) = (ap\ (ap\ (ap\ (ap\ (c\_2Epatricia\_2EBranch$
$A\_27a)\ V5v8)\ V6v9)\ V7v10)\ V8v11)) \wedge (((ap\ (\mathsf{c\_2Epatricia\_2EBRANCH}$
$A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod$
$ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree\ A\_27a)$
$(ty\_2Epatricia\_2Eptree\ A\_27a))))\ V0p)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree\ A\_27a)$
$(ty\_2Epatricia\_2Eptree\ A\_27a)))\ V1m)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$(ty\_2Epatricia\_2Eptree\ A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))$
$(ap\ (ap\ (c\_2Epatricia\_2ELeaf\ A\_27a)\ V9v12)\ V10v13))\ (ap\ (ap\ (c\_2Epatricia\_2ELeaf$
$A\_27a)\ V11v24)\ V12v25))))) = (ap\ (ap\ (ap\ (ap\ (c\_2Epatricia\_2EBranch$
$A\_27a)\ V0p)\ V1m)\ (ap\ (ap\ (c\_2Epatricia\_2ELeaf\ A\_27a)\ V9v12)\ V10v13))$
$(ap\ (ap\ (c\_2Epatricia\_2ELeaf\ A\_27a)\ V11v24)\ V12v25))) \wedge (((ap\ ($
$\mathsf{c\_2Epatricia\_2EBRANCH}\ A\_27a)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}\ ty\_2Enum\_2Enum$
$(ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree$
$A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))))\ V0p)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$ty\_2Enum\_2Enum\ (ty\_2Epair\_2Eprod\ (ty\_2Epatricia\_2Eptree\ A\_27a)$
$(ty\_2Epatricia\_2Eptree\ A\_27a)))\ V1m)\ (ap\ (ap\ (\mathsf{c\_2Epair\_2E\_2C}$
$(ty\_2Epatricia\_2Eptree\ A\_27a)\ (ty\_2Epatricia\_2Eptree\ A\_27a))$
$(ap\ (ap\ (c\_2Epatricia\_2ELeaf\ A\_27a)\ V9v12)\ V10v13))\ (ap\ (ap\ (ap$
$(ap\ (c\_2Epatricia\_2EBranch\ A\_27a)\ V13v26)\ V14v27)\ V15v28)\ V16v29))))) =$
$(ap\ (ap\ (ap\ (ap\ (c\_2Epatricia\_2EBranch\ A\_27a)\ V0p)\ V1m)\ (ap\ (ap\ ($
$c\_2Epatricia\_2ELeaf\ A\_27a)\ V9v12)\ V10v13))\ (ap\ (ap\ (ap\ (ap\ (c\_2Epatricia\_2EBranch$