

# thm\_2Epatricia\_2EMEM\_\_TRAVERSE\_\_INSERT\_\_PTREE (TMcu7UExF23AdEV4gcGtraFuEsGUgTC74iN)

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**Definition 1** We define  $c\_2Emin\_2E\_3D\_3D\_3E$  to be  $\lambda P \in 2.\lambda Q \in 2.inj\_o (p P \Rightarrow p Q)$  of type  $\iota$ .

**Definition 2** We define  $c\_2Emin\_2E\_3D$  to be  $\lambda A.\lambda x \in A.\lambda y \in A.inj\_o (x = y)$  of type  $\iota \Rightarrow \iota$ .

**Definition 3** We define  $c\_2Ebool\_2E\_2T$  to be  $(ap (ap (c\_2Emin\_2E\_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

**Definition 4** We define  $c\_2Ebool\_2E\_21$  to be  $\lambda A\_27a : \iota.(\lambda V0P \in (2^{A\_27a}).(ap (ap (c\_2Emin\_2E\_3D (2^{A\_27a}))$

**Definition 5** We define  $c\_2Ebool\_2E\_5C\_2E\_2F$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

**Definition 6** We define  $c\_2Ebool\_2E\_2F$  to be  $(ap (c\_2Ebool\_2E\_21 2) (\lambda V0t \in 2.V0t))$ .

**Definition 7** We define  $c\_2Ebool\_2E\_7E$  to be  $(\lambda V0t \in 2.(ap (ap c\_2Emin\_2E\_3D\_3D\_3E V0t) c\_2Ebool\_2E\_2F$

Let  $ty\_2Eone\_2Eone : \iota$  be given. Assume the following.

$$nonempty\ ty\_2Eone\_2Eone \tag{1}$$

**Definition 8** We define  $c\_2Emin\_2E\_40$  to be  $\lambda A.\lambda P \in 2^A.if (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge p x))$  of type  $\iota \Rightarrow \iota$ .

**Definition 9** We define  $c\_2Eone\_2Eone$  to be  $(ap (c\_2Emin\_2E\_40 ty\_2Eone\_2Eone) (\lambda V0x \in ty\_2Eone\_2Eone$

**Definition 10** We define  $c\_2Ebool\_2E\_2F\_5C$  to be  $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c\_2Ebool\_2E\_21 2) (\lambda V2t \in 2.V2t))$

Let  $ty\_2Esum\_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Esum\_2Esum\ A0\ A1) \tag{2}$$

Let  $c\_2Esum\_2EABS\_sum : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Esum\_2EABS\_sum\ A\_27a\ A\_27b \in ((ty\_2Esum\_2Esum\ A\_27a\ A\_27b)^{((2^{A\_27b})^{A\_27a})^2}) \tag{3}$$

**Definition 11** We define  $c\_Esum\_2EINR$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0e \in A\_27b. (ap (c\_Esum\_2EABS$   
Let  $ty\_2Eoption\_2Eoption : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow nonempty (ty\_2Eoption\_2Eoption A0) \quad (4)$$

Let  $c\_2Eoption\_2Eoption\_ABS : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow c\_2Eoption\_2Eoption\_ABS A\_27a \in ((ty\_2Eoption\_2Eoption A\_27a)^{(ty\_2Esum\_2Esum A\_27a ty\_2Eone\_2Eone)}) \quad (5)$$

**Definition 12** We define  $c\_2Eoption\_2ENONE$  to be  $\lambda A\_27a : \iota. (ap (c\_2Eoption\_2Eoption\_ABS A\_27a) ($   
Let  $c\_2Eoption\_2EIS\_SOME : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow c\_2Eoption\_2EIS\_SOME A\_27a \in (2^{(ty\_2Eoption\_2Eoption A\_27a)}) \quad (6)$$

Let  $ty\_2Enum\_2Enum : \iota$  be given. Assume the following.

$$nonempty ty\_2Enum\_2Enum \quad (7)$$

Let  $ty\_2Elist\_2Elist : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow nonempty (ty\_2Elist\_2Elist A0) \quad (8)$$

Let  $ty\_2Epatricia\_2Eptree : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0. nonempty A0 \Rightarrow nonempty (ty\_2Epatricia\_2Eptree A0) \quad (9)$$

Let  $c\_2Epatricia\_2ETRAVERSE : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow c\_2Epatricia\_2ETRAVERSE A\_27a \in ((ty\_2Elist\_2Elist ty\_2Enum\_2Enum)^{(ty\_2Epatricia\_2Eptree A\_27a)}) \quad (10)$$

Let  $c\_2Elist\_2ELIST\_TO\_SET : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a. nonempty A\_27a \Rightarrow c\_2Elist\_2ELIST\_TO\_SET A\_27a \in (2^{A\_27a})^{(ty\_2Elist\_2Elist A\_27a)} \quad (11)$$

**Definition 13** We define  $c\_2Ebool\_2EIN$  to be  $\lambda A\_27a : \iota. (\lambda V0x \in A\_27a. (\lambda V1f \in (2^{A\_27a}). (ap V1f V0x)))$

**Definition 14** We define  $c\_2Esum\_2EINL$  to be  $\lambda A\_27a : \iota. \lambda A\_27b : \iota. \lambda V0e \in A\_27a. (ap (c\_2Esum\_2EABS$

**Definition 15** We define  $c\_2Eoption\_2ESOME$  to be  $\lambda A\_27a : \iota. \lambda V0x \in A\_27a. (ap (c\_2Eoption\_2Eoption\_ABS$

**Definition 16** We define  $c\_2Ebool\_2ECOND$  to be  $\lambda A\_27a : \iota. (\lambda V0t \in 2. (\lambda V1t1 \in A\_27a. (\lambda V2t2 \in A\_27a. ($

Let  $ty\_2Epair\_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty\_2Epair\_2Eprod\ A0\ A1) \quad (12)$$

Let  $c\_2Epair\_2EABS\_prod : \iota \Rightarrow \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow \forall A\_27b.nonempty\ A\_27b \Rightarrow c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b \in ((ty\_2Epair\_2Eprod\ A\_27a\ A\_27b)^{(2^{A\_27b})^{A\_27a}}) \quad (13)$$

**Definition 17** We define  $c\_2Epair\_2E\_2C$  to be  $\lambda A\_27a : \iota.\lambda A\_27b : \iota.\lambda V0x \in A\_27a.\lambda V1y \in A\_27b.(ap\ (c\_2Epair\_2EABS\_prod\ A\_27a\ A\_27b)\ x\ y)$

Let  $c\_2Epatricia\_2EADD : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epatricia\_2EADD\ A\_27a \in (((ty\_2Epatricia\_2Eptree\ A\_27a)\ (ty\_2Epair\_2Eprod\ ty\_2Enum\_2Enum\ A\_27a))\ (ty\_2Epatricia\_2Eptree\ A\_27a)) \quad (14)$$

**Definition 18** We define  $c\_2Epatricia\_2EINSERT\_PTREE$  to be  $\lambda V0n \in ty\_2Enum\_2Enum.\lambda V1t \in (ty\_2Epair\_2Eprod\ n\ n)$

Let  $c\_2Epatricia\_2EPEEK : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epatricia\_2EPEEK\ A\_27a \in (((ty\_2Eoption\_2Eoption\ A\_27a)\ (ty\_2Enum\_2Enum)\ (ty\_2Epatricia\_2Eptree\ A\_27a))\ (ty\_2Eoption\_2Eoption\ A\_27a)) \quad (15)$$

Let  $c\_2Epatricia\_2EIS\_PTREE : \iota \Rightarrow \iota$  be given. Assume the following.

$$\forall A\_27a.nonempty\ A\_27a \Rightarrow c\_2Epatricia\_2EIS\_PTREE\ A\_27a \in (2^{(ty\_2Epatricia\_2Eptree\ A\_27a)}) \quad (16)$$

Assume the following.

$$True \quad (17)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p\ V0t1) \Rightarrow (p\ V1t2)) \Rightarrow (((p\ V1t2) \Rightarrow (p\ V0t1)) \Rightarrow ((p\ V0t1) \Leftrightarrow (p\ V1t2)))))) \quad (18)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p\ V0t))) \quad (19)$$

Assume the following.

$$(\forall V0t \in 2.((p\ V0t) \vee (\neg(p\ V0t)))) \quad (20)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)) \wedge (((p\ V0t) \wedge True) \Leftrightarrow (p\ V0t)) \wedge (((False \wedge (p\ V0t)) \Leftrightarrow False) \wedge (((p\ V0t) \wedge False) \Leftrightarrow False) \wedge (((p\ V0t) \wedge (p\ V0t)) \Leftrightarrow (p\ V0t)))))) \quad (21)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \vee (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \vee True) \Leftrightarrow True) \wedge \\
& (((False \vee (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee False) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \vee \\
& (p \ V0t)) \Leftrightarrow (p \ V0t))))))
\end{aligned} \tag{22}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Rightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Rightarrow True) \Leftrightarrow \\
& True) \wedge (((False \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (((p \ V0t) \Rightarrow (p \ V0t)) \Leftrightarrow True) \wedge (( \\
& (p \ V0t) \Rightarrow False) \Leftrightarrow (\neg (p \ V0t))))))
\end{aligned} \tag{23}$$

Assume the following.

$$\begin{aligned}
& ((\forall V0t \in 2.((\neg(\neg(p \ V0t))) \Leftrightarrow (p \ V0t)) \wedge ((\neg True) \Leftrightarrow False) \wedge \\
& ((\neg False) \Leftrightarrow True)))
\end{aligned} \tag{24}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.((V0x = V0x) \Leftrightarrow \\
& True))
\end{aligned} \tag{25}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0x \in A\_27a.(\forall V1y \in \\
& A\_27a.((V0x = V1y) \Leftrightarrow (V1y = V0x))))
\end{aligned} \tag{26}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t \in 2.(((True \Leftrightarrow (p \ V0t)) \Leftrightarrow (p \ V0t)) \wedge (((p \ V0t) \Leftrightarrow True) \Leftrightarrow \\
& (p \ V0t)) \wedge (((False \Leftrightarrow (p \ V0t)) \Leftrightarrow (\neg (p \ V0t))) \wedge (((p \ V0t) \Leftrightarrow False) \Leftrightarrow (\neg( \\
& p \ V0t))))))
\end{aligned} \tag{27}$$

Assume the following.

$$\begin{aligned}
& \forall A\_27a.nonempty \ A\_27a \Rightarrow (\forall V0t1 \in A\_27a.(\forall V1t2 \in \\
& A\_27a.(((ap \ (ap \ (ap \ (c\_2Ebool\_2ECOND \ A\_27a) \ c\_2Ebool\_2ET) \ V0t1) \\
& V1t2) = V0t1) \wedge ((ap \ (ap \ (ap \ (c\_2Ebool\_2ECOND \ A\_27a) \ c\_2Ebool\_2EF) \\
& V0t1) \ V1t2) = V1t2))))))
\end{aligned} \tag{28}$$

Assume the following.

$$\begin{aligned}
& (\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p \ V0t1) \Rightarrow \\
& ((p \ V1t2) \Rightarrow (p \ V2t3))) \Leftrightarrow (((p \ V0t1) \wedge (p \ V1t2)) \Rightarrow (p \ V2t3))))))
\end{aligned} \tag{29}$$

Assume the following.

$$\begin{aligned}
& (\forall V0x \in 2.(\forall V1x\_27 \in 2.(\forall V2y \in 2.(\forall V3y\_27 \in \\
& 2.(((p \ V0x) \Leftrightarrow (p \ V1x\_27)) \wedge ((p \ V1x\_27) \Rightarrow ((p \ V2y) \Leftrightarrow (p \ V3y\_27)))))) \Rightarrow \\
& (((p \ V0x) \Rightarrow (p \ V2y)) \Leftrightarrow ((p \ V1x\_27) \Rightarrow (p \ V3y\_27))))))
\end{aligned} \tag{30}$$

Assume the following.

$$\begin{aligned} \forall A\_27a. \text{nonempty } A\_27a \Rightarrow & ((\forall V0x \in A\_27a. ((p (ap (c\_2Eoption\_2EIS\_SOME \\ A\_27a) (ap (c\_2Eoption\_2ESOME A\_27a) V0x))) \Leftrightarrow True)) \wedge & ((p (ap (c\_2Eoption\_2EIS\_SOME \\ A\_27a) (c\_2Eoption\_2ENONE A\_27a))) \Leftrightarrow False)) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} \forall A\_27a. \text{nonempty } A\_27a \Rightarrow & (\forall V0t \in (ty\_2Epatricia\_2Eptree \\ A\_27a). (\forall V1k \in ty\_2Enum\_2Enum. ((p (ap (c\_2Epatricia\_2EIS\_PTREE \\ A\_27a) V0t)) \Rightarrow & ((p (ap (ap (c\_2Ebool\_2EIN ty\_2Enum\_2Enum) V1k) ( \\ ap (c\_2Elist\_2ELIST\_TO\_SET ty\_2Enum\_2Enum) (ap (c\_2Epatricia\_2ETRAVERSE \\ A\_27a) V0t)))) \Leftrightarrow & (p (ap (c\_2Eoption\_2EIS\_SOME A\_27a) (ap (ap (c\_2Epatricia\_2EPEEK \\ A\_27a) V0t) V1k))))))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in (ty\_2Epatricia\_2Eptree ty\_2Eone\_2Eone). (\forall V1x \in \\ ty\_2Enum\_2Enum. ((p (ap (c\_2Epatricia\_2EIS\_PTREE ty\_2Eone\_2Eone) \\ V0t)) \Rightarrow & (p (ap (c\_2Epatricia\_2EIS\_PTREE ty\_2Eone\_2Eone) (ap ( \\ ap c\_2Epatricia\_2EINSERT\_PTREE V1x) V0t)))))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} (\forall V0t \in (ty\_2Epatricia\_2Eptree ty\_2Eone\_2Eone). (\forall V1k \in \\ ty\_2Enum\_2Enum. (\forall V2j \in ty\_2Enum\_2Enum. ((p (ap (c\_2Epatricia\_2EIS\_PTREE \\ ty\_2Eone\_2Eone) V0t)) \Rightarrow & ((ap (ap (c\_2Epatricia\_2EPEEK ty\_2Eone\_2Eone) \\ (ap (ap c\_2Epatricia\_2EINSERT\_PTREE V1k) V0t)) V2j) = (ap (ap ( \\ ap (c\_2Ebool\_2ECOND (ty\_2Eoption\_2Eoption ty\_2Eone\_2Eone) \\ (ap (ap (c\_2Emin\_2E\_3D ty\_2Enum\_2Enum) V1k) V2j)) (ap (c\_2Eoption\_2ESOME \\ ty\_2Eone\_2Eone) c\_2Eone\_2Eone)) (ap (ap (c\_2Epatricia\_2EPEEK \\ ty\_2Eone\_2Eone) V0t) V2j))))))))) \end{aligned} \quad (34)$$

**Theorem 1**

$$\begin{aligned} (\forall V0t \in (ty\_2Epatricia\_2Eptree ty\_2Eone\_2Eone). (\forall V1x \in \\ ty\_2Enum\_2Enum. (\forall V2h \in ty\_2Enum\_2Enum. ((p (ap (c\_2Epatricia\_2EIS\_PTREE \\ ty\_2Eone\_2Eone) V0t)) \Rightarrow & ((p (ap (ap (c\_2Ebool\_2EIN ty\_2Enum\_2Enum) \\ V1x) (ap (c\_2Elist\_2ELIST\_TO\_SET ty\_2Enum\_2Enum) (ap (c\_2Epatricia\_2ETRAVERSE \\ ty\_2Eone\_2Eone) (ap (ap c\_2Epatricia\_2EINSERT\_PTREE V2h) V0t)))))) \Leftrightarrow & ((V1x = V2h) \vee ((\neg(V1x = V2h)) \wedge (p (ap (ap (c\_2Ebool\_2EIN ty\_2Enum\_2Enum) \\ V1x) (ap (c\_2Elist\_2ELIST\_TO\_SET ty\_2Enum\_2Enum) (ap (c\_2Epatricia\_2ETRAVERSE \\ ty\_2Eone\_2Eone) V0t)))))))))) \end{aligned}$$