

thm_2Epatricia_2EREMOVE_ADD (TMVuPushFQsiGjyHs5KnCzQbWUagfyNA5yg)

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Definition 1 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2.\lambda Q \in 2.inj_o (p P \Rightarrow p Q)$ of type ι .

Definition 2 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 3 We define $c_2Ebool_2E_2T$ to be $(ap (ap (c_2Emin_2E_3D (2^2)) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x))$

Definition 4 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap (ap (c_2Emin_2E_3D (2^{A_27a}))$

Definition 5 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.V2t)))$

Definition 6 We define $c_2Ebool_2E_2F$ to be $(ap (c_2Ebool_2E_21 2) (\lambda V0t \in 2.V0t))$.

Definition 7 We define $c_2Ebool_2E_7E$ to be $(\lambda V0t \in 2.(ap (ap c_2Emin_2E_3D_3D_3E V0t) c_2Ebool_2E_2F$

Let $ty_2Eoption_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty_2Eoption_2Eoption A0) \quad (1)$$

Let $c_2Eoption_2EIS_SOME : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Eoption_2EIS_SOME A_27a \in (2^{(ty_2Eoption_2Eoption A_27a)}) \quad (2)$$

Let $c_2Eoption_2EIS_NONE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Eoption_2EIS_NONE A_27a \in (2^{(ty_2Eoption_2Eoption A_27a)}) \quad (3)$$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Epair_2Eprod A0 A1) \quad (4)$$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty ty_2Eone_2Eone \quad (5)$$

Definition 8 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2.(\lambda V1t2 \in 2.(ap (c_2Ebool_2E_21 2) (\lambda V2t \in 2.))$
Let $ty_2Esum_2Esum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow \forall A1.nonempty A1 \Rightarrow nonempty (ty_2Esum_2Esum A0 A1) \quad (6)$$

Let $c_2Esum_2EABS_sum : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Esum_2EABS_sum A_27a A_27b \in ((ty_2Esum_2Esum A_27a A_27b)^{((2^{A_27b})^{A_27a})^2}) \quad (7)$$

Definition 9 We define c_2Esum_2EINL to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27a.(ap (c_2Esum_2EABS_sum A_27a A_27b) V0e)$
Let $c_2Eoption_2Eoption_ABS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Eoption_2Eoption_ABS A_27a \in ((ty_2Eoption_2Eoption A_27a)^{(ty_2Esum_2Esum A_27a ty_2Eone_2Eone)}) \quad (8)$$

Definition 10 We define $c_2Eoption_2ESOME$ to be $\lambda A_27a : \iota.\lambda V0x \in A_27a.(ap (c_2Eoption_2Eoption_ABS A_27a) V0x)$
Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Epair_2EABS_prod A_27a A_27b \in ((ty_2Epair_2Eprod A_27a A_27b)^{(2^{A_27b})^{A_27a}}) \quad (9)$$

Definition 11 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0x \in A_27a.\lambda V1y \in A_27b.(ap (c_2Epair_2EABS_prod A_27a A_27b) V0x V1y)$
Let $ty_2Epatricia_2Eptree : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty A0 \Rightarrow nonempty (ty_2Epatricia_2Eptree A0) \quad (10)$$

Let $ty_2Eenum_2Eenum : \iota$ be given. Assume the following.

$$nonempty ty_2Eenum_2Eenum \quad (11)$$

Let $c_2Epatricia_2EADD : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Epatricia_2EADD A_27a \in (((ty_2Epatricia_2Eptree A_27a)^{(ty_2Epair_2Eprod ty_2Eenum_2Eenum A_27a)})^{(ty_2Epatricia_2Eptree A_27a)}) \quad (12)$$

Definition 12 We define $c_2Emin_2E_40$ to be $\lambda A.\lambda P \in 2^A.\mathbf{if} (\exists x \in A.p (ap P x)) \mathbf{then} (the (\lambda x.x \in A \wedge P x) x)$
of type $\iota \Rightarrow \iota$.

Definition 13 We define c_2Eone_2Eone to be $(ap (c_2Emin_2E_40 ty_2Eone_2Eone) (\lambda V0x \in ty_2Eone_2Eone.))$

Definition 14 We define c_2Esum_2EINR to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0e \in A_27b.(ap (c_2Esum_2EABS_sum A_27a A_27b) V0e)$

Definition 15 We define $c_2Eoption_2ENONE$ to be $\lambda A_27a : \iota.(ap (c_2Eoption_2Eoption_ABS A_27a) ($

Definition 16 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.($

Let $c_2Epatricia_2EREMOVE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Epatricia_2EREMOVE A_27a \in ((ty_2Epatricia_2Eptree A_27a)^{ty_2Enum_2Enum} (ty_2Epatricia_2Eptree A_27a)) \quad (13)$$

Let $c_2Epatricia_2EPEEK : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Epatricia_2EPEEK A_27a \in (((ty_2Eoption_2Eoption A_27a)^{ty_2Enum_2Enum} (ty_2Epatricia_2Eptree A_27a)) \quad (14)$$

Let $c_2Epatricia_2EIS_PTREE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Epatricia_2EIS_PTREE A_27a \in (2^{(ty_2Epatricia_2Eptree A_27a)}) \quad (15)$$

Assume the following.

$$True \quad (16)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2)))))) \quad (17)$$

Assume the following.

$$(\forall V0t \in 2.(False \Rightarrow (p V0t))) \quad (18)$$

Assume the following.

$$(\forall V0t \in 2.((p V0t) \vee (\neg (p V0t)))) \quad (19)$$

Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow (\forall V0t \in 2.((\forall V1x \in A_27a.(p V0t) \Leftrightarrow (p V0t)))) \quad (20)$$

Assume the following.

$$(\forall V0t \in 2.(((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t)))))) \quad (21)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge ((p V0t) \Rightarrow False) \Leftrightarrow (\neg (p V0t)))))) \quad (22)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (23)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0x \in A.27a.((V0x = V0x) \Leftrightarrow True)) \quad (24)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0x \in A.27a.(\forall V1y \in A.27a.((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (25)$$

Assume the following.

$$(\forall V0t \in 2.(((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (26)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0t1 \in A.27a.(\forall V1t2 \in A.27a.(((ap (ap (ap (c.2Ebool.2ECOND A.27a) c.2Ebool.2ET) V0t1) V1t2) = V0t1) \wedge ((ap (ap (ap (c.2Ebool.2ECOND A.27a) c.2Ebool.2EF) V0t1) V1t2) = V1t2)))) \quad (27)$$

Assume the following.

$$(\forall V0t1 \in 2.(\forall V1t2 \in 2.(\forall V2t3 \in 2.(((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3))))) \quad (28)$$

Assume the following.

$$(\forall V0x \in 2.(\forall V1x.27 \in 2.(\forall V2y \in 2.(\forall V3y.27 \in 2.(((p V0x) \Leftrightarrow (p V1x.27)) \wedge ((p V1x.27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y.27)))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x.27) \Rightarrow (p V3y.27))))) \quad (29)$$

Assume the following.

$$\forall A.27a.nonempty A.27a \Rightarrow (\forall V0P \in 2.(\forall V1Q \in 2.(\forall V2x \in A.27a.(\forall V3x.27 \in A.27a.(\forall V4y \in A.27a.(\forall V5y.27 \in A.27a.(((p V0P) \Leftrightarrow (p V1Q)) \wedge (((p V1Q) \Rightarrow (V2x = V3x.27)) \wedge ((\neg(p V1Q)) \Rightarrow (V4y = V5y.27)))) \Rightarrow ((ap (ap (ap (c.2Ebool.2ECOND A.27a) V0P) V2x) V4y) = (ap (ap (ap (c.2Ebool.2ECOND A.27a) V1Q) V3x.27) V5y.27)))))) \quad (30)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow ((\forall V0t1 \in A_27a. (\forall V1t2 \in \\ & A_27a. ((ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ A_27a)\ c_2Ebool_2ET)\ V0t1)\ \\ & V1t2) = V0t1))) \wedge (\forall V2t1 \in A_27a. (\forall V3t2 \in A_27a. ((ap \\ & (ap\ (ap\ (c_2Ebool_2ECOND\ A_27a)\ c_2Ebool_2EF)\ V2t1)\ V3t2) = V3t2)))))) \end{aligned} \quad (31)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in \\ & A_27a. (((ap\ (c_2Eoption_2ESOME\ A_27a)\ V0x) = (ap\ (c_2Eoption_2ESOME \\ & A_27a)\ V1y)) \Leftrightarrow (V0x = V1y)))))) \end{aligned} \quad (32)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0x \in A_27a. (\neg((c_2Eoption_2ENONE \\ & A_27a) = (ap\ (c_2Eoption_2ESOME\ A_27a)\ V0x)))))) \end{aligned} \quad (33)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0P \in 2. (\forall V1X \in (\\ & ty_2Eoption_2Eoption\ A_27a). (\forall V2x \in A_27a. (((ap\ (ap\ (\\ & ap\ (c_2Ebool_2ECOND\ (ty_2Eoption_2Eoption\ A_27a))\ V0P)\ V1X)\ (\\ & c_2Eoption_2ENONE\ A_27a)) = (c_2Eoption_2ENONE\ A_27a)) \Leftrightarrow ((p\ V0P) \Rightarrow \\ & (p\ (ap\ (c_2Eoption_2EIS_NONE\ A_27a)\ V1X)))) \wedge (((ap\ (ap\ (ap\ (c_2Ebool_2ECOND \\ & (ty_2Eoption_2Eoption\ A_27a))\ V0P)\ (c_2Eoption_2ENONE\ A_27a)) \\ & V1X) = (c_2Eoption_2ENONE\ A_27a)) \Leftrightarrow ((p\ (ap\ (c_2Eoption_2EIS_SOME \\ & A_27a)\ V1X)) \Rightarrow (p\ V0P))) \wedge (((ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ (ty_2Eoption_2Eoption \\ & A_27a))\ V0P)\ V1X)\ (c_2Eoption_2ENONE\ A_27a)) = (ap\ (c_2Eoption_2ESOME \\ & A_27a)\ V2x)) \Leftrightarrow ((p\ V0P) \wedge (V1X = (ap\ (c_2Eoption_2ESOME\ A_27a)\ V2x)))) \wedge \\ & (((ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ (ty_2Eoption_2Eoption\ A_27a)) \\ & V0P)\ (c_2Eoption_2ENONE\ A_27a))\ V1X) = (ap\ (c_2Eoption_2ESOME \\ & A_27a)\ V2x)) \Leftrightarrow ((\neg(p\ V0P)) \wedge (V1X = (ap\ (c_2Eoption_2ESOME\ A_27a)\ \\ & V2x)))))))))) \end{aligned} \quad (34)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in (ty_2Epatricia_2Eptree \\ & A_27a). (\forall V1x \in (ty_2Epair_2Eprod\ ty_2Enum_2Enum\ A_27a). \\ & ((p\ (ap\ (c_2Epatricia_2EIS_PTREE\ A_27a)\ V0t)) \Rightarrow (p\ (ap\ (c_2Epatricia_2EIS_PTREE \\ & A_27a)\ (ap\ (ap\ (c_2Epatricia_2EADD\ A_27a)\ V0t)\ V1x)))))) \end{aligned} \quad (35)$$

Assume the following.

$$\begin{aligned} & \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in (ty_2Epatricia_2Eptree \\ & A_27a). (\forall V1k \in ty_2Enum_2Enum. ((p\ (ap\ (c_2Epatricia_2EIS_PTREE \\ & A_27a)\ V0t)) \Rightarrow (p\ (ap\ (c_2Epatricia_2EIS_PTREE\ A_27a)\ (ap\ (ap\ (\\ & c_2Epatricia_2EREMOVE\ A_27a)\ V0t)\ V1k)))))) \end{aligned} \quad (36)$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in (ty_2Epatricia_2Eptree \\
& \quad A_27a).(\forall V1k \in ty_2Enum_2Enum.(\forall V2d \in A_27a.(\forall V3j \in \\
& \quad ty_2Enum_2Enum.((p\ (ap\ (c_2Epatricia_2EIS_PTREE\ A_27a)\ V0t)) \Rightarrow \\
& \quad ((ap\ (ap\ (c_2Epatricia_2EPEEK\ A_27a)\ (ap\ (ap\ (c_2Epatricia_2EADD \\
& \quad A_27a)\ V0t)\ (ap\ (ap\ (c_2Epair_2E_2C\ ty_2Enum_2Enum\ A_27a)\ V1k) \\
& \quad V2d)))\ V3j) = (ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ (ty_2Eoption_2Eoption \\
& \quad A_27a))\ (ap\ (ap\ (c_2Emin_2E_3D\ ty_2Enum_2Enum)\ V1k)\ V3j))\ (ap\ (\\
& \quad c_2Eoption_2ESOME\ A_27a)\ V2d))\ (ap\ (ap\ (c_2Epatricia_2EPEEK\ A_27a) \\
& \quad V0t)\ V3j)))))))))
\end{aligned} \tag{37}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in (ty_2Epatricia_2Eptree \\
& \quad A_27a).(\forall V1k \in ty_2Enum_2Enum.(\forall V2j \in ty_2Enum_2Enum. \\
& \quad ((p\ (ap\ (c_2Epatricia_2EIS_PTREE\ A_27a)\ V0t)) \Rightarrow ((ap\ (ap\ (c_2Epatricia_2EPEEK \\
& \quad A_27a)\ (ap\ (ap\ (c_2Epatricia_2EREMOVE\ A_27a)\ V0t)\ V1k))\ V2j) = (\\
& \quad ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ (ty_2Eoption_2Eoption\ A_27a))\ (ap \\
& \quad (ap\ (c_2Emin_2E_3D\ ty_2Enum_2Enum)\ V1k)\ V2j))\ (c_2Eoption_2ENONE \\
& \quad A_27a))\ (ap\ (ap\ (c_2Epatricia_2EPEEK\ A_27a)\ V0t)\ V2j)))))))))
\end{aligned} \tag{38}$$

Assume the following.

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t1 \in (ty_2Epatricia_2Eptree \\
& \quad A_27a).(\forall V1t2 \in (ty_2Epatricia_2Eptree\ A_27a).(((p\ (ap \\
& \quad (c_2Epatricia_2EIS_PTREE\ A_27a)\ V0t1)) \wedge (p\ (ap\ (c_2Epatricia_2EIS_PTREE \\
& \quad A_27a)\ V1t2))) \Rightarrow ((\forall V2k \in ty_2Enum_2Enum.((ap\ (ap\ (c_2Epatricia_2EPEEK \\
& \quad A_27a)\ V0t1)\ V2k) = (ap\ (ap\ (c_2Epatricia_2EPEEK\ A_27a)\ V1t2)\ V2k))) \Leftrightarrow \\
& \quad (V0t1 = V1t2))))))
\end{aligned} \tag{39}$$

Theorem 1

$$\begin{aligned}
& \forall A_27a.nonempty\ A_27a \Rightarrow (\forall V0t \in (ty_2Epatricia_2Eptree \\
& \quad A_27a).(\forall V1k \in ty_2Enum_2Enum.(\forall V2d \in A_27a.(\forall V3j \in \\
& \quad ty_2Enum_2Enum.((p\ (ap\ (c_2Epatricia_2EIS_PTREE\ A_27a)\ V0t)) \Rightarrow \\
& \quad ((ap\ (ap\ (c_2Epatricia_2EREMOVE\ A_27a)\ (ap\ (ap\ (c_2Epatricia_2EADD \\
& \quad A_27a)\ V0t)\ (ap\ (ap\ (c_2Epair_2E_2C\ ty_2Enum_2Enum\ A_27a)\ V1k) \\
& \quad V2d)))\ V3j) = (ap\ (ap\ (ap\ (c_2Ebool_2ECOND\ (ty_2Epatricia_2Eptree \\
& \quad A_27a))\ (ap\ (ap\ (c_2Emin_2E_3D\ ty_2Enum_2Enum)\ V1k)\ V3j))\ (ap\ (\\
& \quad ap\ (c_2Epatricia_2EREMOVE\ A_27a)\ V0t)\ V3j))\ (ap\ (ap\ (c_2Epatricia_2EADD \\
& \quad A_27a)\ (ap\ (ap\ (c_2Epatricia_2EREMOVE\ A_27a)\ V0t)\ V3j))\ (ap\ (ap \\
& \quad (c_2Epair_2E_2C\ ty_2Enum_2Enum\ A_27a)\ V1k)\ V2d)))))))))
\end{aligned}$$