

thm_2Epatricia_2EUNION__PTREE__COMM
 (TMbfGGsch-
 GiZCjc1mUc9EPqyBHNnqjQvXHJ)

October 26, 2020

Definition 1 We define $c_2Emin_2E_3D$ to be $\lambda A.\lambda x \in A.\lambda y \in A.inj_o (x = y)$ of type $\iota \Rightarrow \iota$.

Definition 2 We define $c_2Ebool_2E_ET$ to be $(ap (ap (c_2Emin_2E_3D (2^2))) (\lambda V0x \in 2.V0x)) (\lambda V1x \in 2.V1x)$

Let $ty_2Eone_2Eone : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eone_2Eone \tag{1}$$

Let $ty_2Eoption_2Eoption : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Eoption_2Eoption\ A0) \tag{2}$$

Let $ty_2Eenum_2Eenum : \iota$ be given. Assume the following.

$$nonempty\ ty_2Eenum_2Eenum \tag{3}$$

Let $ty_2Epatricia_2Eptree : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Epatricia_2Eptree\ A0) \tag{4}$$

Let $c_2Epatricia_2EPEEK : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Epatricia_2EPEEK\ A_27a \in (((ty_2Eoption_2Eoption\ A_27a)^{ty_2Eenum_2Eenum})^{(ty_2Epatricia_2Eptree\ A_27a)}) \tag{5}$$

Let $c_2Eoption_2EIS_SOME : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Eoption_2EIS_SOME\ A_27a \in (2^{(ty_2Eoption_2Eoption\ A_27a)}) \tag{6}$$

Definition 3 We define $c_2Ebool_2E_21$ to be $\lambda A_27a : \iota. (\lambda V0P \in (2^{A_27a}). (ap (ap (c_2Emin_2E_3D (2^{A_27a})))$

Definition 4 We define $c_2Epatricia_2EIN_PTREE$ to be $\lambda V0n \in ty_2Eenum_2Eenum.\lambda V1t \in (ty_2Epatricia_2Eptree$

Let $ty_2Elist_2Elist : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow nonempty\ (ty_2Elist_2Elist\ A0) \quad (7)$$

Let $c_2Epatricia_2ETRAVERSE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Epatricia_2ETRAVERSE\ A_27a \in ((ty_2Elist_2Elist\ ty_2Enum_2Enum)^{(ty_2Epatricia_2Eptree\ A_27a)}) \quad (8)$$

Let $c_2Elist_2ELIST_TO_SET : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Elist_2ELIST_TO_SET\ A_27a \in ((2^{A_27a})^{(ty_2Elist_2Elist\ A_27a)}) \quad (9)$$

Definition 5 We define $c_2Epatricia_2ENUMSET_OF_PTREE$ to be $\lambda V0t \in (ty_2Epatricia_2Eptree\ ty_2Elist_2Elist)$

Let $c_2Ebool_2EARB : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Ebool_2EARB\ A_27a \in A_27a \quad (10)$$

Let $c_2Epred_set_2ECHOICE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Epred_set_2ECHOICE\ A_27a \in (A_27a^{(2^{A_27a})}) \quad (11)$$

Definition 6 We define c_2Ebool_2EF to be $(ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V0t \in 2.V0t))$.

Definition 7 We define $c_2Epred_set_2EEMPTY$ to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a.c_2Ebool_2EF)$.

Definition 8 We define c_2Ebool_2EIN to be $\lambda A_27a : \iota. (\lambda V0x \in A_27a. (\lambda V1f \in (2^{A_27a}). (ap\ V1f\ V0x)))$

Definition 9 We define $c_2Emin_2E_3D_3D_3E$ to be $\lambda P \in 2. \lambda Q \in 2. inj_o\ (p\ P \Rightarrow p\ Q)$ of type ι .

Definition 10 We define $c_2Ebool_2E_5C_2F$ to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in 2.V2t))))$

Definition 11 We define $c_2Ebool_2E_2F_5C$ to be $(\lambda V0t1 \in 2. (\lambda V1t2 \in 2. (ap\ (c_2Ebool_2E_21\ 2)\ (\lambda V2t \in 2.V2t))))$

Let $ty_2Epair_2Eprod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A0.nonempty\ A0 \Rightarrow \forall A1.nonempty\ A1 \Rightarrow nonempty\ (ty_2Epair_2Eprod\ A0\ A1) \quad (12)$$

Let $c_2Epair_2EABS_prod : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epair_2EABS_prod\ A_27a\ A_27b \in ((ty_2Epair_2Eprod\ A_27a\ A_27b)^{(2^{A_27b})^{A_27a}}) \quad (13)$$

Definition 12 We define $c_2Epair_2E_2C$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0x \in A_27a. \lambda V1y \in A_27b. (ap\ (c_2Emin_2E_3D_3D_3E\ (c_2Ebool_2EIN\ A_27a)\ (c_2Ebool_2EIN\ A_27b)))$

Let $c_2Epred_set_2EGSPEC : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow \forall A_27b.nonempty\ A_27b \Rightarrow c_2Epred_set_2EGSPEC\ A_27a\ A_27b \in ((2^{A_27a})^{((ty_2Epair_2Eprod\ A_27a\ 2)^{A_27b})}) \quad (14)$$

Definition 13 We define $c_2Epred_set_2EINSERT$ to be $\lambda A_27a : \iota.\lambda V0x \in A_27a.\lambda V1s \in (2^{A_27a}).(ap\ (c_2Ebool_2E7E\ V0t)\ (ap\ (c_2Emin_2E3D_3D_3E\ V0t)\ c_2Ebool_2E21\ V1s))$

Definition 14 We define c_2Ebool_2E7E to be $(\lambda V0t \in 2.(ap\ (ap\ c_2Emin_2E3D_3D_3E\ V0t)\ c_2Ebool_2E21\ V0t))$

Definition 15 We define $c_2Epred_set_2EDIFF$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1t \in (2^{A_27a}).(ap\ (c_2Ebool_2E21\ V0s)\ (ap\ (c_2Ebool_2E21\ V1t)\ (c_2Ebool_2E21\ V0s)))$

Definition 16 We define $c_2Epred_set_2EDELETE$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).\lambda V1x \in A_27a.(ap\ (c_2Ebool_2E21\ V0s)\ (ap\ (c_2Ebool_2E21\ V1x)\ (c_2Ebool_2E21\ V0s)))$

Definition 17 We define $c_2Epred_set_2EREST$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).(ap\ (ap\ (c_2Epred_set_2EREST\ V0s)\ (c_2Ebool_2E21\ V0s)))$

Let $c_2Elist_2ECONS : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Elist_2ECONS\ A_27a \in (((ty_2Elist_2Elist\ A_27a)^{(ty_2Elist_2Elist\ A_27a)})^{A_27a}) \quad (15)$$

Let $c_2Elist_2ENIL : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty\ A_27a \Rightarrow c_2Elist_2ENIL\ A_27a \in (ty_2Elist_2Elist\ A_27a) \quad (16)$$

Definition 18 We define c_2Emin_2E40 to be $\lambda A.\lambda P \in 2^A.\mathbf{if}\ (\exists x \in A.p\ (ap\ P\ x))\ \mathbf{then}\ (the\ (\lambda x.x \in A)\ p)$ of type $\iota \Rightarrow \iota$.

Definition 19 We define c_2Ebool_2ECOND to be $\lambda A_27a : \iota.(\lambda V0t \in 2.(\lambda V1t1 \in A_27a.(\lambda V2t2 \in A_27a.(ap\ (c_2Ebool_2E21\ V1t1)\ (c_2Ebool_2E21\ V2t2))))$

Definition 20 We define $c_2Epred_set_2EFINITE$ to be $\lambda A_27a : \iota.\lambda V0s \in (2^{A_27a}).(ap\ (c_2Ebool_2E21\ V0s)\ (ap\ (c_2Ebool_2E21\ V0s)\ (c_2Ebool_2E21\ V0s)))$

Definition 21 We define $c_2Ecombin_2EK$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.(\lambda V0x \in A_27a.(\lambda V1y \in A_27b.V0x))$

Definition 22 We define $c_2Ecombin_2ES$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda A_27c : \iota.(\lambda V0f \in ((A_27c^{A_27b})^{A_27a}))$

Definition 23 We define $c_2Ecombin_2EI$ to be $\lambda A_27a : \iota.(ap\ (ap\ (c_2Ecombin_2ES\ A_27a\ (A_27a^{A_27a})\ A_27a)\ (c_2Ebool_2E21\ V0f)))$

Definition 24 We define c_2Ebool_2E3F to be $\lambda A_27a : \iota.(\lambda V0P \in (2^{A_27a}).(ap\ V0P\ (ap\ (c_2Emin_2E40\ V0P)\ (c_2Ebool_2E21\ V0P))))$

Definition 25 We define $c_2Erelation_2EWF$ to be $\lambda A_27a : \iota.\lambda V0R \in ((2^{A_27a})^{A_27a}).(ap\ (c_2Ebool_2E21\ V0R)\ (ap\ (c_2Ebool_2E21\ V0R)\ (c_2Ebool_2E21\ V0R))))$

Definition 26 We define $c_2Erelation_2ERESTRICT$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0f \in (A_27b^{A_27a}).\lambda V1M \in (A_27a^{A_27a}).(ap\ (c_2Ebool_2E21\ V0f)\ (ap\ (c_2Ebool_2E21\ V1M)\ (c_2Ebool_2E21\ V0f))))$

Definition 27 We define $c_2Erelation_2ETC$ to be $\lambda A_27a : \iota.\lambda V0R \in ((2^{A_27a})^{A_27a}).\lambda V1a \in A_27a.\lambda V2b \in A_27a.(ap\ (c_2Ebool_2E21\ V0R)\ (ap\ (c_2Ebool_2E21\ V1a)\ (ap\ (c_2Ebool_2E21\ V2b)\ (c_2Ebool_2E21\ V0R))))$

Definition 28 We define $c_2Erelation_2Eapprox$ to be $\lambda A_27a : \iota.\lambda A_27b : \iota.\lambda V0R \in ((2^{A_27a})^{A_27a}).\lambda V1M \in (A_27a^{A_27a}).(ap\ (c_2Ebool_2E21\ V0R)\ (ap\ (c_2Ebool_2E21\ V1M)\ (c_2Ebool_2E21\ V0R))))$

Definition 29 We define $c_2Erelation_2Ethe_fun$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0R \in ((2^{A_27a})^{A_27a}). \lambda V1M$

Definition 30 We define $c_2Erelation_2EWFREC$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda V0R \in ((2^{A_27a})^{A_27a}). \lambda V1M$

Definition 31 We define $c_2Elist_2ESET_TO_LIST$ to be $\lambda A_27a : \iota. (ap (ap (c_2Erelation_2EWFREC (2^{A_27a})$

Definition 32 We define c_2Eone_2Eone to be $(ap (c_2Emin_2E_40 ty_2Eone_2Eone) (\lambda V0x \in ty_2Eone_2Eone$

Let $c_2Epatricia_2EADD : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Epatricia_2EADD A_27a \in (((ty_2Epatricia_2Eptree A_27a)(ty_2Epair_2Eprod ty_2Eenum_2Eenum A_27a)(ty_2Epatricia_2Eptree A_27a))$$

$$(17)$$

Definition 33 We define $c_2Epatricia_2EINSERT_PTREE$ to be $\lambda V0n \in ty_2Eenum_2Eenum. \lambda V1t \in (ty_2Eenum$

Definition 34 We define $c_2Ecombin_2Eec$ to be $\lambda A_27a : \iota. \lambda A_27b : \iota. \lambda A_27c : \iota. (\lambda V0f \in ((A_27c)^{A_27b})^{A_27a}$

Let $c_2Elist_2Efoldl : \iota \Rightarrow \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow \forall A_27b.nonempty A_27b \Rightarrow c_2Elist_2Efoldl A_27a A_27b \in (((A_27b)^{(ty_2Elist_2Elist A_27a)})^{A_27b})^{((A_27b)^{A_27a})^{A_27b}}$$

$$(18)$$

Definition 35 We define $c_2Epatricia_2EPTREE_OF_NUMSET$ to be $\lambda V0t \in (ty_2Epatricia_2Eptree ty_2Eenum$

Definition 36 We define $c_2Epatricia_2EUNION_PTREE$ to be $\lambda V0t1 \in (ty_2Epatricia_2Eptree ty_2Eone$

Let $c_2Epatricia_2EIS_PTREE : \iota \Rightarrow \iota$ be given. Assume the following.

$$\forall A_27a.nonempty A_27a \Rightarrow c_2Epatricia_2EIS_PTREE A_27a \in (2^{(ty_2Epatricia_2Eptree A_27a)})$$

$$(19)$$

Assume the following.

$$True$$

$$(20)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (((p V0t1) \Rightarrow (p V1t2)) \Rightarrow (((p V1t2) \Rightarrow (p V0t1)) \Rightarrow ((p V0t1) \Leftrightarrow (p V1t2))))))$$

$$(21)$$

Assume the following.

$$(\forall V0t \in 2. (((True \wedge (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \wedge True) \Leftrightarrow (p V0t)) \wedge (((False \wedge (p V0t)) \Leftrightarrow False) \wedge (((p V0t) \wedge False) \Leftrightarrow False) \wedge (((p V0t) \wedge (p V0t)) \Leftrightarrow (p V0t))))))$$

$$(22)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Rightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Rightarrow True) \Leftrightarrow True) \wedge (((False \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow (p V0t)) \Leftrightarrow True) \wedge (((p V0t) \Rightarrow False) \Leftrightarrow (\neg (p V0t))))))$$

$$(23)$$

Assume the following.

$$((\forall V0t \in 2.((\neg(\neg(p V0t))) \Leftrightarrow (p V0t))) \wedge (((\neg True) \Leftrightarrow False) \wedge ((\neg False) \Leftrightarrow True))) \quad (24)$$

Assume the following.

$$\forall A_27a.nonempty \ A_27a \Rightarrow (\forall V0x \in A_27a. (\forall V1y \in A_27a. ((V0x = V1y) \Leftrightarrow (V1y = V0x)))) \quad (25)$$

Assume the following.

$$(\forall V0t \in 2. (((True \Leftrightarrow (p V0t)) \Leftrightarrow (p V0t)) \wedge (((p V0t) \Leftrightarrow True) \Leftrightarrow (p V0t)) \wedge (((False \Leftrightarrow (p V0t)) \Leftrightarrow (\neg(p V0t))) \wedge (((p V0t) \Leftrightarrow False) \Leftrightarrow (\neg(p V0t))))) \quad (26)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p V0A) \vee (p V1B) \wedge (p V2C)) \Leftrightarrow (((p V0A) \vee (p V1B)) \wedge ((p V0A) \vee (p V2C))))) \quad (27)$$

Assume the following.

$$(\forall V0A \in 2. (\forall V1B \in 2. (\forall V2C \in 2. (((p V1B) \wedge (p V2C)) \vee (p V0A)) \Leftrightarrow (((p V1B) \vee (p V0A)) \wedge ((p V2C) \vee (p V0A))))) \quad (28)$$

Assume the following.

$$(\forall V0t1 \in 2. (\forall V1t2 \in 2. (\forall V2t3 \in 2. (((p V0t1) \Rightarrow ((p V1t2) \Rightarrow (p V2t3))) \Leftrightarrow (((p V0t1) \wedge (p V1t2)) \Rightarrow (p V2t3)))) \quad (29)$$

Assume the following.

$$(\forall V0x \in 2. (\forall V1x_27 \in 2. (\forall V2y \in 2. (\forall V3y_27 \in 2. (((p V0x) \Leftrightarrow (p V1x_27)) \wedge ((p V1x_27) \Rightarrow ((p V2y) \Leftrightarrow (p V3y_27))) \Rightarrow (((p V0x) \Rightarrow (p V2y)) \Leftrightarrow ((p V1x_27) \Rightarrow (p V3y_27))))) \quad (30)$$

Assume the following.

$$(\forall V0t1 \in (ty_2Epatricia_2Eptree \ ty_2Eone_2Eone). (\forall V1t2 \in (ty_2Epatricia_2Eptree \ ty_2Eone_2Eone). (((p (ap (c_2Epatricia_2EIS_PTREE ty_2Eone_2Eone) V0t1)) \wedge (p (ap (c_2Epatricia_2EIS_PTREE ty_2Eone_2Eone) V1t2))) \Rightarrow ((V0t1 = V1t2) \Leftrightarrow (\forall V2x \in ty_2Enum_2Enum. ((p (ap (ap c_2Epatricia_2EIN_PTREE V2x) V0t1)) \Leftrightarrow (p (ap (ap c_2Epatricia_2EIN_PTREE V2x) V1t2))))))) \quad (31)$$

Assume the following.

$$(\forall V0t1 \in (ty_2Epatricia_2Eptree \ ty_2Eone_2Eone). (\forall V1t2 \in (ty_2Epatricia_2Eptree \ ty_2Eone_2Eone). (\forall V2n \in ty_2Enum_2Enum. (((p (ap (c_2Epatricia_2EIS_PTREE ty_2Eone_2Eone) V0t1)) \wedge (p (ap (c_2Epatricia_2EIS_PTREE ty_2Eone_2Eone) V1t2))) \Rightarrow ((p (ap (ap c_2Epatricia_2EIN_PTREE V2n) (ap (ap c_2Epatricia_2EUNION_PTREE V0t1) V1t2))) \Leftrightarrow ((p (ap (ap c_2Epatricia_2EIN_PTREE V2n) V0t1)) \vee (p (ap (ap c_2Epatricia_2EIN_PTREE V2n) V1t2))))))) \quad (32)$$

Assume the following.

$$\begin{aligned}
& (\forall V0t1 \in (ty_2Epatricia_2Eptree\ ty_2Eone_2Eone).(\forall V1t2 \in \\
& (ty_2Epatricia_2Eptree\ ty_2Eone_2Eone).(((p\ (ap\ (c_2Epatricia_2EIS_PTREE \\
& ty_2Eone_2Eone)\ V0t1)) \wedge (p\ (ap\ (c_2Epatricia_2EIS_PTREE\ ty_2Eone_2Eone) \\
& V1t2))) \Rightarrow (p\ (ap\ (c_2Epatricia_2EIS_PTREE\ ty_2Eone_2Eone)\ (ap \\
& (ap\ c_2Epatricia_2EUNION_PTREE\ V0t1)\ V1t2))))))
\end{aligned} \tag{33}$$

Assume the following.

$$(\forall V0t \in 2.((\neg(\neg(p\ V0t))) \Leftrightarrow (p\ V0t))) \tag{34}$$

Assume the following.

$$(\forall V0A \in 2.((p\ V0A) \Rightarrow ((\neg(p\ V0A)) \Rightarrow False))) \tag{35}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(p\ V0A) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\
& ((p\ V0A) \Rightarrow False) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))))
\end{aligned} \tag{36}$$

Assume the following.

$$\begin{aligned}
& (\forall V0A \in 2.(\forall V1B \in 2.(((\neg((\neg(\neg(p\ V0A)) \vee (p\ V1B))) \Rightarrow False) \Leftrightarrow \\
& ((p\ V0A) \Rightarrow ((\neg(p\ V1B)) \Rightarrow False))))))
\end{aligned} \tag{37}$$

Assume the following.

$$(\forall V0A \in 2.(((\neg(p\ V0A)) \Rightarrow False) \Rightarrow (((p\ V0A) \Rightarrow False) \Rightarrow False))) \tag{38}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\
& (p\ V1q) \Leftrightarrow (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((p\ V1q) \vee (p\ V2r))) \wedge (((p\ V0p) \vee ((\neg(\\
& p\ V2r)) \vee (\neg(p\ V1q)))))) \wedge (((p\ V1q) \vee ((\neg(p\ V2r)) \vee (\neg(p\ V0p)))))) \wedge ((p\ V2r) \vee \\
& ((\neg(p\ V1q)) \vee (\neg(p\ V0p))))))))))
\end{aligned} \tag{39}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\
& (p\ V1q) \wedge (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee ((\neg(p\ V1q)) \vee (\neg(p\ V2r)))))) \wedge (((p\ V1q) \vee \\
& (\neg(p\ V0p))) \wedge ((p\ V2r) \vee (\neg(p\ V0p))))))))))
\end{aligned} \tag{40}$$

Assume the following.

$$\begin{aligned}
& (\forall V0p \in 2.(\forall V1q \in 2.(\forall V2r \in 2.(((p\ V0p) \Leftrightarrow (\\
& (p\ V1q) \vee (p\ V2r))) \Leftrightarrow (((p\ V0p) \vee (\neg(p\ V1q))) \wedge (((p\ V0p) \vee (\neg(p\ V2r)))))) \wedge \\
& ((p\ V1q) \vee ((p\ V2r) \vee (\neg(p\ V0p))))))))))
\end{aligned} \tag{41}$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\forall V2r \in 2. (((p V0p) \Leftrightarrow (p V1q) \Rightarrow (p V2r)) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (((p V0p) \vee \neg(p V2r))) \wedge (\neg(p V1q) \vee ((p V2r) \vee \neg(p V0p)))))))))) \quad (42)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (((p V0p) \Leftrightarrow \neg(p V1q)) \Leftrightarrow (((p V0p) \vee (p V1q)) \wedge (\neg(p V1q) \vee \neg(p V0p)))))) \quad (43)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow (p V0p))) \quad (44)$$

Assume the following.

$$(\forall V0p \in 2. (\forall V1q \in 2. (\neg((p V0p) \Rightarrow (p V1q))) \Rightarrow \neg(p V1q))) \quad (45)$$

Theorem 1

$$(\forall V0t1 \in (ty_2Epatricia_2Eptree\ ty_2Eone_2Eone). (\forall V1t2 \in (ty_2Epatricia_2Eptree\ ty_2Eone_2Eone). (((p (ap (c_2Epatricia_2EIS_PTREE\ ty_2Eone_2Eone)\ V0t1)) \wedge (p (ap (c_2Epatricia_2EIS_PTREE\ ty_2Eone_2Eone)\ V1t2))) \Rightarrow ((ap (ap\ c_2Epatricia_2EUNION_PTREE\ V0t1)\ V1t2) = (ap (ap\ c_2Epatricia_2EUNION_PTREE\ V1t2)\ V0t1))))))$$